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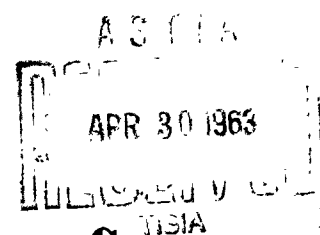
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UNIVERSITY OF ILLINOIS - URBANA, ILLINOIS

**RATIONAL FUNCTION APPROXIMATION
OF POLYNOMIALS WITH
EQUIRIPPLE ERROR**

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**COORDINATED SCIENCE LABORATORY
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ABSTRACT

A method has been proposed whereby any polynomial can be approximated in an equiripple manner by a rational function. The properties and form of this rational function are discussed. Several examples are used to illustrate the theory. Most of these examples are chosen so that the approximating rational function can be identified as a network function part; in particular, the group delay has been given special emphasis. The ideal group delay vs. frequency characteristic of filter is a constant. This type of group delay is approximated in an equiripple manner.

In addition, a numerical scheme is proposed such that from a given crude equiripple approximation a more exact solution can be obtained. An example used to illustrate this approach is the problem of compensating the non-constant group delay characteristics of a sharp cutoff low pass filter.

ERRATA

1. page v, Figure 5 caption should be tangential not tanbential.
2. page v, Figure 10 should be part not port.
3. page v, Figure 16 should be the not th.
4. page vi, Figure 21 delete the x.
5. page 14, equation 2.14b should read $P_l + \epsilon - 2\epsilon \frac{Q_+}{Q_+ - Q_-}$.
6. page 15, Table 1, column III, row entitled "Multi constant of Q_+ " the entry should be $\frac{a_n - c_o}{\epsilon}$.
7. page 24, paragraph 2, line 5, insert function between "rational which".
8. page 26, second figure instead of odd should be N odd.
9. page 28, bottom of page constraint c should read $R_e G_N(n^2) = (-1)^n$ for $n = 0, \pm 1, \pm 2, \dots, \pm (N-1)$.
10. page 29, second paragraph from solution in first sentence, "is" should be replaced by it.
11. page 30, label top curve N even and bottom curve N odd.
12. page 33, label top curve N even and bottom curve N odd.
13. page 45, equation 3.37 states $k = 0, 1$ and should be $k = 0, 2$.
14. page 51, lower 1/3 of the page under heading b, second sentence "resiude" should be spelled residue.
15. page 57, part h should be guess not guesse.
16. page 57, second to the last sentence delete of the appendix.
17. page 60, Table 3 the heading of the last column should be y in %.
18. page 69, equation 4.1 should be written as

$$\text{error} = \pm \epsilon_1 \epsilon \left(\ln \epsilon_2 / \epsilon_1 \right) (\omega / \omega_1)^2$$
19. page 72, last sentence of the figure caption approximate should be inserted between "configurations and unity".

1. HISTORICAL BACKGROUND AND STATEMENT OF THE PROBLEM

1.1 Status of Equiripple Rational Functions as Used in Filter Theory

The use of equiripple polynomials (such polynomials are more commonly known as Chebyshev polynomials) in filter theory has become classical knowledge.^{30,31} Briefly, these polynomials approximate zero in the normalized interval $(-1, 1)$ in an equiripple or Chebyshev fashion. Furthermore, consistent with the degree of the polynomial, the maximum number of peak deviations is obtained.⁸ To attain this maximum number of deviations, all of the polynomial's degrees of freedom are utilized (the multiplying constant is not considered a degree of freedom).

The elliptic function as used in equiripple pass and stopband filter approximations is also of a classical nature.⁶ Instead of a polynomial, however, the elliptic function is rational, i.e., a ratio of two polynomials. Within the interval for which zero is being approximated, this rational function behaves like the Chebyshev polynomials. Outside of this region a difference occurs in that the rational function also approximates infinity in an equiripple manner. Again the number of maximum deviations is the largest number possible for the chosen degree of the rational function.²¹

To obtain some freedom and more versatility in equiripple approximations, Sharpe²⁵, following closely the methods of Bernstein⁵, developed a rational function with equiripple properties within a prescribed interval. Furthermore, the poles of this function were free to be chosen on the imaginary frequency axis (outside the approximating interval) and on the real frequency axis of the complex plane. Bennett⁴, through the use of conformal transformations, and Helman¹⁶, by extending the definitions of the Chebyshev polynomials, succeeded in developing a rational function with arbitrary complex plane pole locations. Such functional forms are of a general type, the Chebyshev polynomials and elliptic rational functions being special cases. In addition to the above, many authors have contributed to the area of equiripple functions. An extensive list of some authors is given in the literature.³¹

1.2 Status of Constant Group Delay Approximations

A complete history and a mention of all who have contributed to the area of group delay approximations would be a volume in itself. Thus, only a few of the most important will be mentioned.

The first classical contribution in the area of group delay approximation was made with the appearance of Thompson's²⁸ paper on the maximally flat group delay. Later, Storch²⁶ showed that the polynomial obtained by Thompson was a class of Bessel polynomials, hence the common name "Bessel filter". Utilizing the great storehouse of knowledge involving Bessel functions, Storch gave this solution a very elegant treatment.

As the Bessel filter gives an excellent approximation near the origin, but suffers somewhat for higher frequencies, a logical extension was an attempt to approximate the group delay in the large.²² Several authors, among the most important of which are Guillemin¹⁴, Kuh¹⁸, and Darlington¹⁰, have succeeded in approximating the group delay in this manner to give near equiripple group delay characteristics over a larger frequency interval. Briefly, their methods of approach were (1) choosing an equally spaced pole distribution parallel to the real frequency axis in the complex plane, (2) utilizing a potential analog approach to determine the best pole distribution on an ellipse in the complex plane, and (3) employing a Chebyshev polynomial series to approximate a linear phase. At this point another approximation can be mentioned which would be classified as an approximation near the origin. It is the so-called Gaussian magnitude approximation due to Dishal.¹¹ The point of view used here departs from that of the previous authors in that the phase is attacked indirectly. Instead of considering the phase, the magnitude function is made to approximate the Gaussian error function by truncating its Taylor series expansion. The final result is a monotonically decreasing group delay which is relatively constant within the passband.

Following closely the work of Helman¹⁶, Ulbrich and Piloty²⁹ utilized Bennett's⁴ conformal transformation to formulate a set of non-linear simultaneous equations that when solved would give an equiripple group delay. The set was then solved using iterative techniques on a digital computer. The main disadvantage to their published results is that the magnitude of the ripple is an absolute number when, as shown in the next section, the time domain error, or dispersion, is proportional (in a first order approximation) to the fractional or percentage group delay ripple. Recognizing this shortcoming, the author employed the rational function developed in Chapter 2 to formulate a different set of non-linear simultaneous equations which were also solved using an iterative scheme on a digital computer. The author's work was just completed on equiripple group delay, however, when Abele² published similar results.

1.3 Ideal Low Pass Filters with Near Constant Group Delay

One objective of this work is to obtain several realizable transfer functions with an equiripple group delay characteristic. Since such systems are used primarily for the transmission and filtering of pulses, a time domain error in the impulse response of a filter due to the group delay ripple would be of interest. If two reasonably valid simplifying assumptions are made, an error estimate is easily obtained using the method first described by Wheeler.³² These assumptions are (1) the group delay ripple is cosinusoidal, and (2) the group delay is approximately constant over the frequency spectrum for which the filter does not give a large attenuation.

Consider a transfer function $H(j\omega)$ which has a group delay $\tau(\omega)$ of the form

$$\tau(\omega) = \tau_0 (1 - (-1)^m \epsilon \cos \frac{m\pi\omega}{\omega_0}) \quad (1.1)$$

where

- τ_0 = average delay within the filter's passband
- ϵ = the peak fractional group delay deviation
- m = the number of half cycles of ripple the group delay possesses in the interval $(0, \omega_0)$
- ω_0 = normalizing constant.

The group delay for such an idealization is illustrated in Figure 1.

Utilizing the definition of the group delay

$$\tau(\omega) = - \frac{dB(\omega)}{d\omega} \quad (1.2)$$

the phase of $H(j\omega)$ is found to be

$$\beta(\omega) = -\tau_0 \left[\omega - (-1)^m \epsilon \frac{\omega_0}{m\pi} \sin \frac{m\pi\omega}{\omega_0} \right] \quad (1.3)$$

Papoulis²³ has shown, using Wheeler's method³², that the impulse response $i(t)$ of the transfer function $H(j\omega)$ with the phase characteristic of (1.3) can be put in the form:

$$i(t) = \sum_{k=-\infty}^{\infty} J_k(\delta) h\left(t - \frac{km\pi}{\omega_0}\right) \quad (1.4)$$

where

$h(t)$ is the inverse of $|H(j\omega)| e^{-j\omega\tau_o}$

J_k is the Bessel function of the first kind, and

$$\delta = \frac{\epsilon \tau_o \omega_o}{m\pi} (-1)^m.$$

If δ is very small compared to one, (1.4) reduces to

$$i(t) \approx h(t) + \frac{\delta}{2} h\left(t - \frac{m\pi}{\omega_o}\right) - \frac{\delta}{2} h\left(t + \frac{m\pi}{\omega_o}\right). \quad (1.5)$$

For a lumped-element system the phase of the transfer function approaches $n\pi/2$ as ω tends to infinity, where n is the number of poles less the number of finite zeros. Hence, the area under the group delay curve from $\omega = 0$ to $\omega = \infty$ is $m\pi/2$. If the fractional part η of this area follows the boundary of the area from $\omega = 0$ to $\omega = \omega_o$ of Figure 1, then

$$\tau_o = \frac{m\pi}{2\omega_o} \eta$$

$$\delta = (-1)^m \frac{n\eta\epsilon}{2m}$$

and (1.5) can be written as

$$i(t) \approx h(t) + (-1)^m \frac{n\eta\epsilon}{4m} h\left(t - \frac{2\tau_o}{\eta} \frac{m}{n}\right) - (-1)^m \frac{n\eta\epsilon}{4m} h\left(t + \frac{2\tau_o}{\eta} \frac{m}{n}\right). \quad (1.6)$$

A typical impulse response will then appear as shown in Figure 2.

From the foregoing analysis, several conclusions can be drawn. Some of the more important are:

1. The output pulse is dispersed in the sense that the time bandwidth product is increased because of the non-constant group delay.
2. The leading "echo" appears to give an anticipatory feature to the filter. In a physical system, however, this echo would in effect cancel with the main pulse to prevent such a phenomenon.

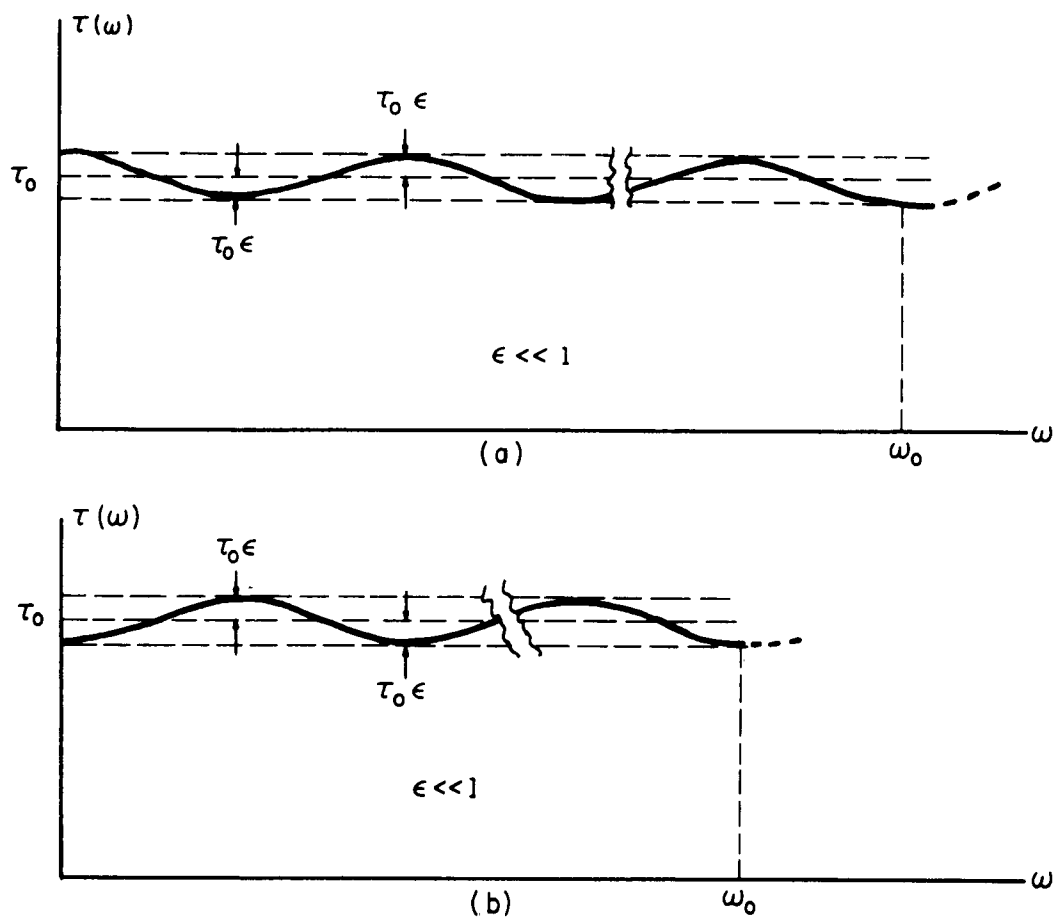


Figure 1. Idealized equiripple group delay with m (a) odd and (b) even

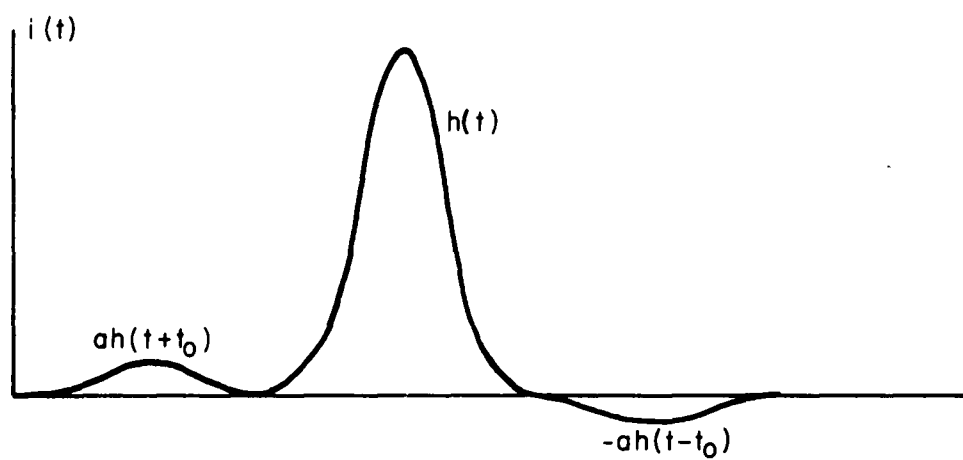


Figure 2. Impulse response of a filter with the idealized group delay of Figure 1

3. The magnitude of the two main distortion terms, centered at $\tau_0 \pm 2\tau_0 m/n \eta$, is proportional to the fractional ripple ϵ .
4. The distortion or error terms are proportional to n/m , i.e., the ratio of the number of $\pi/2$ multiples of phase at $\omega = \infty$ to the number of half-cycles of ripple.
5. The dispersion terms are proportional to the constant η .

1.4 Statement of the Problem

The Chebyshev rational function of Helman and Bennett is a very practical and useful form as applied to network problems. However, general and useful as it is, it does possess some basic limitations. Two of these are:

1. If a polynomial more general than a constant is to be approximated in the Chebyshev sense a serious difficulty is encountered. For example: using Bennett's conformal transformation, suppose a linear term is to be approximated with a rational function of a chosen degree. The end result demands that the poles have to be chosen in such a manner that a zero will appear in the proper location to nullify the effect of a particular pole at the same location. Because of its form, this would indeed be a difficult task.
2. Some problems require a rational function approximation of a polynomial such that the number of ripples is less than the maximum possible number. Again the currently available Chebyshev rational function is not adaptable to such a situation.

The problem is thus to develop a closed form rational function which is general enough to solve many of the problems solvable with the existing Chebyshev rational function and, in addition, be sufficiently flexible to overcome the above stated difficulties. Once such a function is developed there exist many practical network problems to which it can be applied; some solved, and some unsolved. Thus, after the main problem has been studied, several of the solved and unsolved varieties can be investigated with the purpose of exploiting the potentials of another tool in today's existing body of knowledge. Special emphasis will be given to group delay problems, as it

has been only recently that equiripple solutions to this problem have appeared^{29,2}, and another method of attack would be of value.

As general as the rational function is which overcomes the two stated difficulties of the Bennett and Helman forms, it is not difficult to think of equiripple or weighted equiripple problems which are not easily adapted to fit any equiripple rational function. With this in mind, a numerical scheme is suggested and illustrated to solve some of the existing group delay problems which may be encountered.

2. THE DEVELOPMENT OF A PROCEDURE FOR APPROXIMATING POLYNOMIALS WITH RATIONAL FUNCTIONS TO GIVE AN EQUI RIPPLE ERROR

Many problems arise wherein a rational function is desired which approximates a polynomial. Often this polynomial is of a low degree; indeed, most practical problems demand that it be nothing more than a constant. The problem undertaken in this chapter is thus the determination of some properties and how to construct such a rational function.

2.1 The General Case

The rational function

$$R_{nm}(x) \equiv \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^m b_i x^i} \quad (2.1)$$

of given numerator and denominator degrees n and m respectively is to be found which approximates the given polynomial.

$$P_\ell(x) \equiv \sum_{i=0}^{\ell} c_i x^i \quad (2.2)$$

of degree ℓ in an equiripple manner in the interval $\alpha \leq x \leq \beta$. The constants a_i , b_i , and c_i are all real, and, without loss of generality, b_m is assumed to be unity. If R_{nm} is to approximate P_ℓ in $[\alpha, \beta]$ with an equiripple error of peak deviation ϵ then

$$P_\ell - \epsilon \leq R_{nm} \leq P_\ell + \epsilon \quad \text{for} \quad \alpha \leq x \leq \beta.$$

The number of times the equality is satisfied as x takes on all values between α and β will be called n_e . An upper bound on n_e will be determined subsequently.

As an intermediate step in determining the numerator $A_n(x)$ and denominator $B_m(x)$ or the rational function $R_{nm}(x)$, the error functions $E_+(x)$ and $E_-(x)$ will be defined. From these, some useful properties will be deduced.

$$E_+(x) = R_{nm}(x) - [P_\ell(x) + \epsilon] \quad (2.3)$$

$$E_-(x) = R_{nm}(x) - [P_\ell(x) - \epsilon] ,$$

or, in terms of A_n and B_m

$$E_+ = \frac{A_n - B_m (P_\ell + \epsilon)}{B_m} \quad (2.4)$$

$$E_- = \frac{A_n - B_m (P_\ell - \epsilon)}{B_m} .$$

For an arbitrary example, the functions R_{nm} , P_ℓ , E_+ and E_- are shown in Figure 3.

Except for one rare case, the numerator polynomials of E_+ and E_- are of degree k where

$$k = \text{the larger of } \begin{cases} \ell + m \\ n \end{cases} .$$

The one exception occurs when $n = m + \ell$ ($n \neq 0$) and $a_n = c_\ell$. For this combination, the degree of E_+ and E_- is $k - 1$. If $n = m + \ell$ with $n = 0$ the problem is trivial since only constants are involved. Thus, it can be concluded that the number of zeros of E_+ and E_- is k (for the above noted exception $k - 1$). For R_{nm} to approximate P_ℓ in an equiripple manner with the largest possible number of maximum deviations of $\pm \epsilon$ all zeros of E_+ and E_- are required to be real. Furthermore, because of the tangency at the internal zeros these will be double. The two zeros at the ends of the approximation interval will be simple.

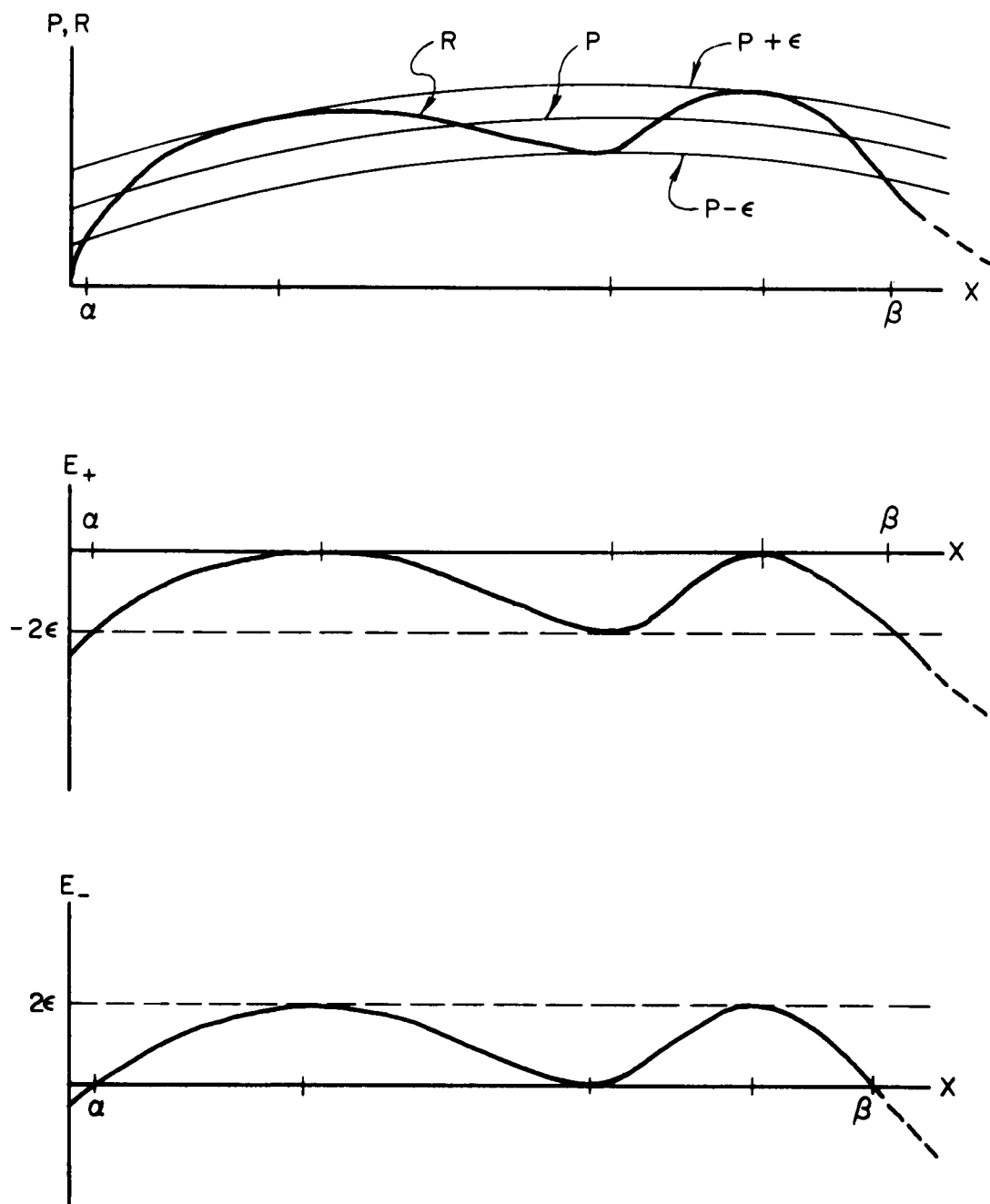


Figure 3. The functions R , P , E_+ and E_- of an arbitrary example

The maximum number, n_e , of $\pm \epsilon$ deviations is now easily determined. Knowing that the total number of zeros involved in E_+ and E_- is $2k$ and all but two of these are double, the number of extrema points is

$$n_e = \frac{2k - 2}{2} + 2 = k + 1 .$$

Unless stated otherwise in the following, the k zeros of E_+ and E_- will be assumed to be (1) real, (2) all double except the two at each end of the interval, and (3) interlaced, i.e., in going from $x = \alpha$ to $x = \beta$ a zero of E_+ (E_-) will be followed by a zero of E_- (E_+), etc.

Making the definitions

$$\begin{aligned} \text{numerator of } E_+ &= Q_+ = A_n - (P_\ell + \epsilon)B_m \\ \text{numerator of } E_- &= Q_- = A_n - (P_\ell - \epsilon)B_m , \end{aligned} \tag{2.5}$$

it is possible to distinguish five different cases which will exist depending upon the degrees ℓ , m , and n . These cases are listed as follows:

$$\text{Case Ia: } n > \ell + m ; \quad \ell \geq 1 ; \quad k = n$$

$$\begin{aligned} Q_+ &= a_n \prod_{i=1}^k (x - g_i) \\ Q_- &= a_n \prod_{i=1}^k (x - h_i) \end{aligned} \tag{2.6}$$

$$\text{Case IB: } n = \ell + m ; \quad \ell \geq 1 ; \quad k = n$$

$$\begin{aligned} Q_+ &= (a_n - c_\ell) \prod_{i=1}^{k-} (x - g_i) \\ Q_- &= (a_n - c_\ell) \prod_{i=1}^k (x - h_i) \end{aligned} \tag{2.7}$$

Case II: $n > m$; $\ell = 0$; $k = n$

$$\begin{aligned} Q_+ &= a_n \prod_{i=1}^k (x - g_i) \\ Q_- &= a_n \prod_{i=1}^k (x - h_i) \end{aligned} \quad (2.8)$$

Case III: $n = m$; $\ell = 0$; $k = n$

$$\begin{aligned} Q_+ &= (a_n - c_0 - \epsilon) \prod_{i=1}^k (x - g_i) \\ Q_- &= (a_n - c_0 + \epsilon) \prod_{i=1}^k (x - h_i) \end{aligned} \quad (2.9)$$

Case IV: $n < m$; $\ell = 0$; $k = m$

$$\begin{aligned} Q_+ &= - (c_0 + \epsilon) \prod_{i=1}^k (x - g_i) \\ Q_- &= - (c_0 - \epsilon) \prod_{i=1}^k (x - h_i) \end{aligned} \quad (2.10)$$

Case V: $n < m + \ell$; $\ell \geq 1$; $k = m + \ell$

$$\begin{aligned} Q_+ &= - c_\ell \prod_{i=1}^k (x - g_i) \\ Q_- &= - c_\ell \prod_{i=1}^k (x - h_i) \end{aligned} \quad (2.11)$$

Two important differences to note in the above cases are (1) whether or not the constant multiplying factors of Q_- and Q_+ are the same and (2) the value of k .

Knowing Q_- and Q_+ in terms of the multiplying constant and the points of maximum deviations, it is a simple matter to solve for A_n and B_m from (2.5). The results are

$$A_n = \frac{(P_\ell + \epsilon) Q_- - (P_\ell - \epsilon) Q_+}{2\epsilon} \quad (2.12)$$

$$B_m = \frac{Q_- - Q_+}{2\epsilon} .$$

Hence, R_{nm} is

$$R_{nm} \equiv \frac{A_n}{B_m} = \frac{(P_\ell + \epsilon) Q_- - (P_\ell - \epsilon) Q_+}{Q_- - Q_+} \quad (2.13)$$

which can also be rearranged to fit into the forms

$$R_{nm} = P_\ell + \epsilon \frac{Q_- + Q_+}{Q_- - Q_+} \quad (2.14a)$$

$$= P_\ell + \epsilon \frac{Q_+}{Q_+ - Q_-} \quad (2.14b)$$

$$R_{nm} = P_\ell - \epsilon + 2\epsilon \frac{Q_-}{Q_- - Q_+} \quad (2.14c)$$

The number of arbitrary constants for a given case can now be determined. The solution will be given in detail for Case Ia. The results are tabulated in Table I for the remaining cases.

The defining relationships of Case Ia are $n > \ell + m$; $\ell \geq 1$. From these it is evident that $k = n$, and $n_e = n + 1$. The following information is obtained by an inspection of (2.1), (2.2), (2.6), and (2.13).

Case	I _a	I _b	II	III	IV	V
Defining Conditions	$n > \ell + m$ $\ell \geq 1$	$n = \ell + m$ $\ell \geq 1$	$n > m$ $\ell = 0$	$n = m$ $\ell = 0$	$n < m$ $\ell = 0$	$n < \ell + m$ $\ell \geq 1$
k	n	n	n	n	m	$\ell + m$
n _e	n+1	n+1	n+1	n+1	m+1	$\ell + m + 1$
n ₃	n-1	n-1	n-1	n	m	$\ell + m - 1$
n ₄	$n + \ell - 1$	$n + \ell - 1$	n	n	m-1	$m + 2\ell - 1$
n ₅	$n + \ell + 3$	$n + \ell + 3$	n+3	n+4	m+3	$2\ell + m + 3$
n ₆	$\ell - 1$	$\ell - 1$	0	0	m-1-n	$m + 2\ell - 1 - n$
n ₇	$n - 1 - m$	$n - 1 - m$	$n - 1 - m$	n-m	0	$\ell + 1$
n ₈	$\ell + n - m - 2$	$\ell + n - m - 2$	$n - 1 - m$	n-m	m-n-1	$m + 3\ell - n - 2$
n ₉	m+5	m+5	m+4	m+4	n+4	$n - \ell + 5$
Mult Cons- tant of Q ₊	a _n	$a - c_{\ell}$	a _n	$a - c_0 + \epsilon$	$-(c_0 + \epsilon)$	$-c_{\ell}$
Mult Cons- tant of Q ₋	a _n	$a - c_{\ell}$	a _n	$a - c_0 + \epsilon$	$-(c_0 - \epsilon)$	$-c_{\ell}$

Table 1. Tabulation of various constants and degrees for the five cases of rational functions

$$n_3 \equiv \text{degree of } Q_+ - Q_- = n - 1$$

$$n_4 \equiv \text{degree of } Q_+ (P_\ell - \epsilon) = n + \ell - 1$$

$$\begin{aligned} n_5 &\equiv \text{total number of constants} \\ &= 1 (\epsilon) + n_e (\text{number of error extrema}) \\ &\quad + \ell + 1 (\text{number of constants in } P_\ell) \\ &\quad + n_x (n_x = 1 \text{ for case 2, zero otherwise}) \\ &= n + \ell + 3 \end{aligned}$$

$$\begin{aligned} n_6 &\equiv \text{number of numerator coefficients to be zero so that } n \text{ is the} \\ &\quad \text{desired integer} \\ &= n_4 - n \\ &= \ell - 1 \end{aligned}$$

$$\begin{aligned} n_7 &\equiv \text{number of denominator coefficients to be zero so that } m \text{ is the} \\ &\quad \text{desired integer} \\ &= n_3 - m \\ &= n - 1 - m \end{aligned}$$

$$\begin{aligned} n_8 &\equiv \text{number of constraints which have to be satisfied} \\ &= n_6 + n_7 \\ &= \ell + n - m - 2 \end{aligned}$$

$$\begin{aligned} n_9 &\equiv \text{number of arbitrary constants} \\ &= n_9 - n_8 \\ &= m + 5 \end{aligned}$$

A word of caution should be inserted at this point. The arbitrary constants are not unlimited in value. Rather, there are definite bounds placed upon their range, this range being defined by the particular problem being solved.

An example will now be given to illustrate the foregoing theory. A rational equiripple approximation to the polynomial $P_1 = x$ is desired in the interval $(1, 10)$. The error extrema, ϵ , is to be $1/2$, and the degree of the numerator and denominator is to be 4. A solution is as follows:

Since $\ell = 1$ and $m = n = 4$, the inequality $n < \ell + m$ is satisfied and the problem fits Case V, hence,

- (a) the maximum number of extreme points = 6
- (b) the number of arbitrary constants = 8
- (c) the number of constraints = 1.

Five of the arbitrary constants have been specified (ϵ , c_0 , c_1 , α , β). Since the number of extrema — of which two are specified and three are arbitrary — is 6, one extreme point will be defined when the others are chosen. The approximation will be similar to the illustration of Figure 4.

From (2.13), the general form of R_{nm} is

$$R_{44} = \frac{Q_- (P_\ell + \epsilon) - Q_+ (P_\ell - \epsilon)}{Q_- - Q_+} \quad (2.15)$$

Using the notation shown in Figure 4

$$\begin{aligned} Q_+ &= - (x - 1) (x - x_2)^2 (x - x_4)^2 \\ Q_- &= - (x - 10) (x - x_3)^2 (x - x_1)^2 \end{aligned} \quad (2.16)$$

The requirement that $n = 4$ places a restriction on the choice of the set (x_1, \dots, x_4) . Equating the coefficient of x^5 to zero, or setting

$$x_4 - x_3 + x_2 - x_1 - 4 = 0 \quad (2.17)$$

satisfies the demand.

An arbitrary choice of x_1 , x_2 , and x_3 is 3, 4, and 8 respectively. This choice makes $x_4 = 11$, but this is outside the approximation range and so cannot be used. Similarly, a choice of 1, 7, and 8 for x_1 , x_2 , and x_3 respectively requires $x_4 = 6$. Again this cannot be used because x_4 must be greater than x_3 and less than 10. A choice of 3, 5, and 7, however, gives a realizable x_4 , namely 9. With the last set of x_i 's the rational function reduces to

$$R_{44} = \frac{1.5x^4 - 34x^3 + 337x^2 - 1578x + 3217.5}{x^4 - 28x^3 + 294x^2 - 1356x + 2385} \quad (2.18)$$

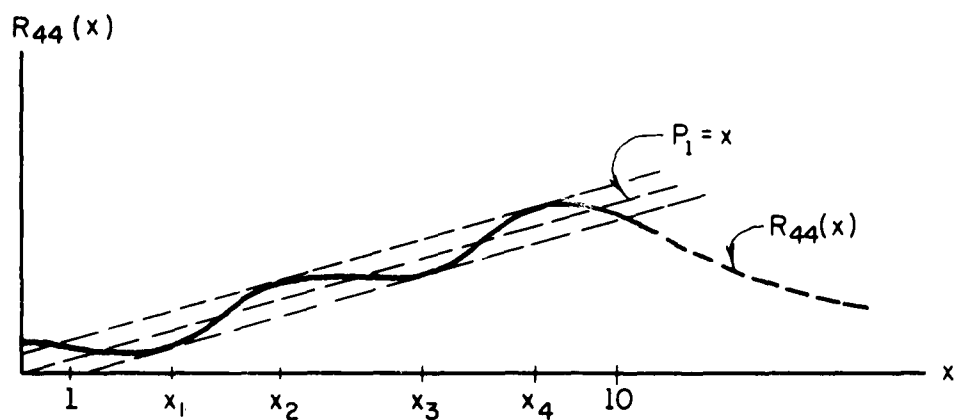


Figure 4. The polynomial $P_1 = x$ approximated in an equiripple manner with R_{44}

Before continuing on to another topic, one more detail needs to be investigated. This detail concerns the possibility of the existence of another rational function \bar{R}_{nm} which (1) passes through $P_\ell \pm \epsilon$ at the ends of the approximating interval (α, β) , (2) is still tangent to P_ℓ and passes through $P_\ell \pm \epsilon$ at the internal points $x_1, x_2, \dots, x_{n_e-2}$, and (3) yet, doesn't intersect P_ℓ within (α, β) . Seemingly, such a rational function (Figure 5) is easily sketched. From such a sketch, however, it is observed that outside the interval (α, β) the function \bar{R}_{nm} must satisfy

$$|P_\ell - \bar{R}_{nm}| < \epsilon \quad (2.19)$$

or \bar{R}_{nm} will intersect with $P_\ell \pm \epsilon$. Such an intersection is impossible, however, because the degree of the numerator of E_+ is not large enough to permit any more intersections. As ϵ, α , and β are arbitrary, it is concluded that the inequality (2.19) cannot hold. Hence, \bar{R}_{nm} does not exist.

2.2 Even Rationals Approximating Even Polynomials

The case under consideration is a special, but important, case of Section 1. Because both R_{nm} and P_ℓ are even, only one half of $(-\beta, \beta)$ need be considered. The approximation is carried out as though P_ℓ was being approximated in $(0, \beta)$. Then at every zero of E_+ and E_- the factor $(x - x_1)$ is replaced by $(x^2 - x_1^2)$. Several examples of this type of approximation are given in Chapter 3.

2.3 Odd Rationals Approximating Odd Polynomials

Again this is a special case of Section 1. Under the assumption that R_{nm} and P_ℓ are odd, the following conclusions can be drawn:

a. From the arbitrary odd approximation shown in Figure 6 it is noted that

$$K_- Q_- (x) = -K_+ Q_+ (-x)$$

where K_+ and K_- are multiplying constants.

b. Since P_ℓ is odd, $\ell \neq 0$; hence from Table 1 it is seen that the multiplying constants of Q_- and Q_+ are equal. Thus

$$K_- = K_+.$$

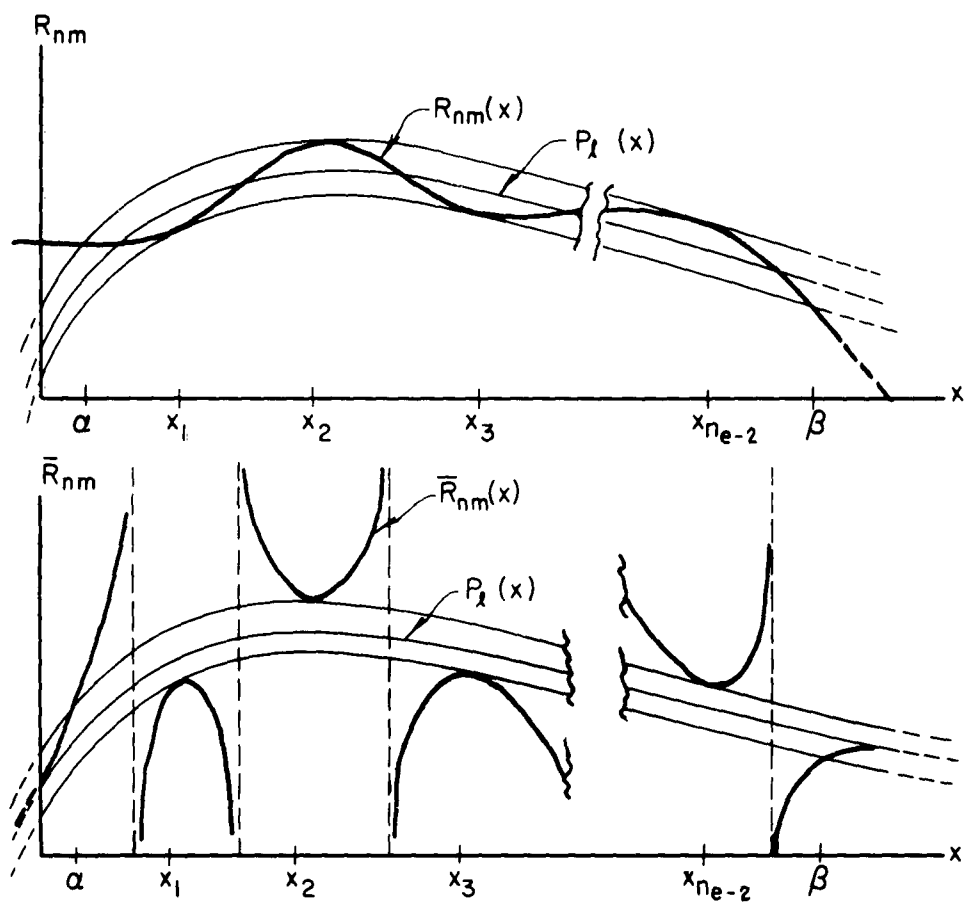


Figure 5. The rational function R_{nm} and \bar{R}_{nm} which both go through $P_l \pm \epsilon$ at $x = \alpha, \beta$ and are tangential to P_l , and pass through $P_l \pm \epsilon$ at the internal points $x_1, x_2, \dots, x_{ne-2}$

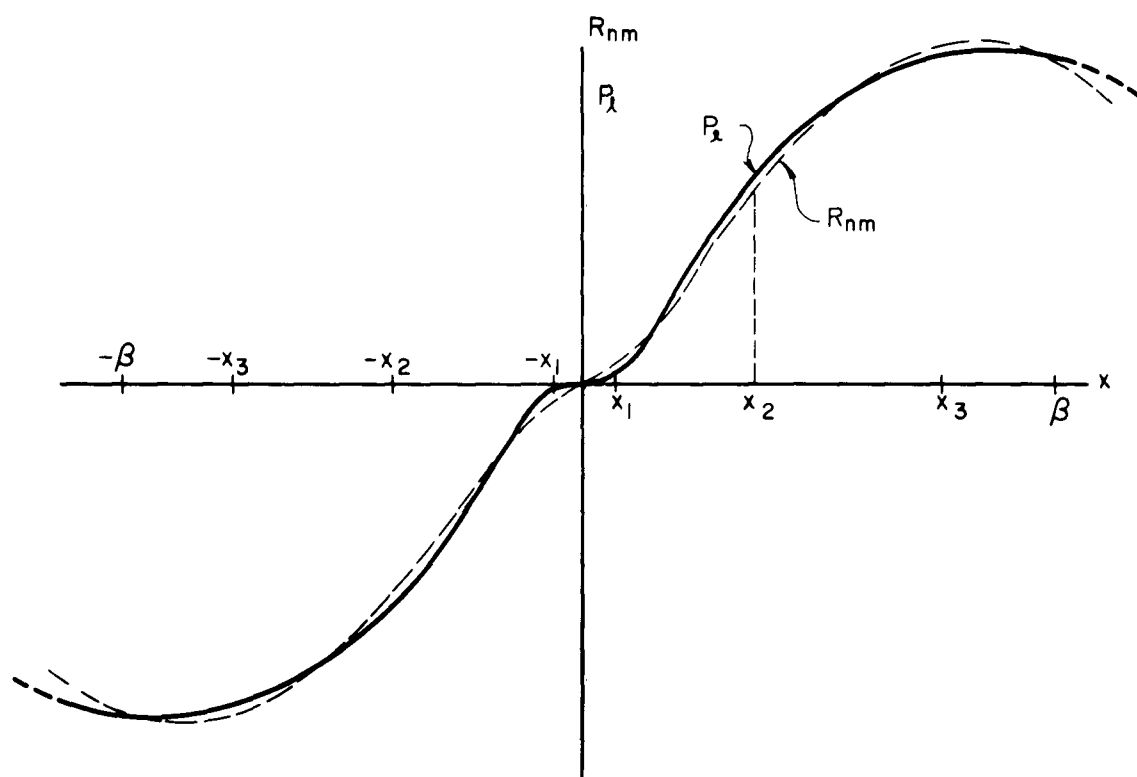


Figure 6. An arbitrary odd polynomial P approximated by an odd rational function R_{nm}

c. Finally, it is observed that

$$P_{\ell}(x) + \epsilon = - [P_{\ell}(-x) - \epsilon] .$$

Putting all three conclusions into (2.13)

$$\begin{aligned} R_{nm} &= \frac{Q_+(-x) [P_{\ell}(-x) - \epsilon] - Q_+(x) [P_{\ell}(x) - \epsilon]}{-Q_+(-x) - Q_+(x)} \\ &= \frac{\text{Odd} \{ Q_+(x) [P_{\ell}(x) - \epsilon] \}}{\text{Ev} [Q_+(x)]} \end{aligned} \quad (2.20)$$

where Odd and Ev are operators which are defined to take the odd and even parts, respectively, of what follows.

Examples of this type approximation are given in Chapter 3.

2.4 Observations and Summary

In this chapter a rational function has been developed which will approximate any polynomial in an equiripple manner. The method has several desirable features. Some of these are:

- a. There is a large amount of freedom available in the form of arbitrary constants, for fitting the rational function to a wide range of problems.
- b. The rational function need not be restricted to approximating only constants as does the rational function of Helman and Bennett.
- c. Though not mentioned explicitly, this rational function need not approximate P_{ℓ} with the maximum number of ripples. If fewer ripples are desired, instead of using the factor $(x - x_1)$ in E_{\pm} where x_1 is positive and real, the term x_1 can be made negative. Indeed, quadratic factors with complex roots can be used, if desired, to decrease the number of ripples. As the constants within these factors are arbitrary, the rational function is given a more arbitrary character. An example of this type is given in Chapter 3.

Some limitations on R_{nm} as developed in this chapter are:

- a. The denominator is the difference of two polynomials and is not in factored form.
- b. In developing a rational function R_{nm} which approximates a high order polynomial or has several zeros at infinity, several non-linear simultaneous equations will have to be solved.

The above limitations, however, are overcome to a large extent by the ever increasing availability of high speed digital computers.

3. APPLICATIONS OF THE EQUIRIPPLE ERROR RATIONAL FUNCTION OF CHAPTER 2 TO NETWORK PROBLEMS

The well known theorem of Weierstrass says that any function continuous in the interval (a, b) may be approximated uniformly by polynomials in this interval.⁹ In general this requires a polynomial of infinite degree. Fortunately, there exists a large class of problems which do not require an approximation of such high degree. Indeed, in many problems the desired function is itself a low order polynomial. Such is the case with many of the network function parts.

A network function is a rational function of the complex variable s , which may in turn be broken up into several parts, many of which are themselves rational in the real variable ω . Among these are the real part, the imaginary part, the magnitude squared function, and the group delay function. This chapter will thus be devoted to finding a rational which approximates a low order polynomial and yet can be identified with one of the above network function parts. This in turn may be physically realized by conventional techniques if desired.³⁰

3.1 Real Part Approximation

The real part of a network function $H(s)$ is a rational function of the real variable ω . That this is the case can be shown very easily by considering the network function.

$$H(s) = \frac{M_1 + N_1}{M_2 + N_2} \quad (3.1)$$

where M_2 and N_2 are the even and odd parts, respectively, of a Hurwitz polynomial in the complex variable $s = \sigma + j\omega$. Similarly, M_1 and N_1 are the even and odd parts of any polynomial. From (3.1) the even part

$$\text{Ev } H(s) = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} \quad (3.2)$$

is easily found. By inspection this is also a rational function. On the real frequency axis $s = j\omega$, and the even part becomes the real part

$$\operatorname{Re} H(j\omega) = \left. \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} \right|_{s=j\omega} \quad (3.3)$$

which is rational in the real variable ω .

If a rational function is to approximate a polynomial and yet be identified as a "positive real" real part it must not only approximate the polynomial, but satisfy the following properties:³⁰

- a. possess no poles on the imaginary axis
- b. the poles and zeros must have quadrantal symmetry
- c. if the denominator of $H(s)$ is of degree t , then the denominator of $\operatorname{Ev} H(s)$ is of degree $2t$. The degree, $2r$, of the numerator of $\operatorname{Ev} H(s)$ must satisfy the inequality

$$0 \leq 2r \leq 2t$$

- d. imaginary axis zeros of $\operatorname{Ev} H(s)$ must be of even multiplicity.

If $H(s)$ is not positive real \underline{d} and \underline{a} can be relaxed.

Several examples will now be given wherein a low order polynomial is approximated by the rational function of Chapter 2, and yet is manipulated to satisfy some or all of the above positive real properties.

Example (3.1)

A rational function is to be obtained which satisfies all of the above positive real properties and in addition is to

- a. approximate the degenerate polynomial "unity" with a tolerance of unity
- b. have a maximum number of extreme points which are separated by equal frequency intervals
- c. have a denominator of degree $2N$
- d. approach the constant value of 4 for high frequencies.

An illustration of the desired function is shown in Figure 7 for N even and N odd.

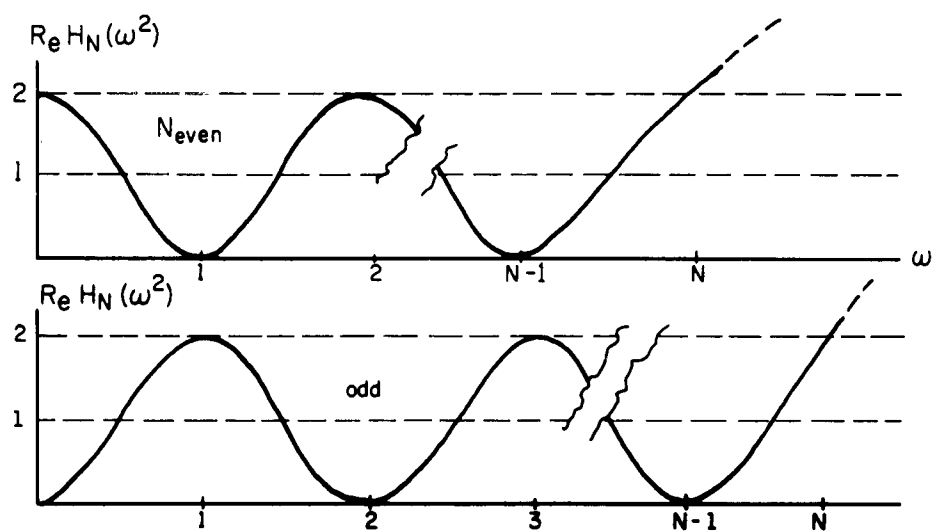


Figure 7. A sketch of the desired real part. The function is given for positive frequencies only as it is even

Solution:

It is first observed that $\ell = 0$, and $m = n$ so that Case III of Chapter 2 applies to this example. Furthermore, R_{nm} and P_ℓ are both even so the corresponding simplifications can be used. For the half-interval $(0, N)$

$$\left. \begin{aligned} Q_+ &= (a_{2N} - 1 - \epsilon) (\omega - N) \omega \prod_{i=1}^{\frac{N-2}{2}} [\omega - (N - 2i)]^2 \\ Q_- &= (a_{2N} - 1 + \epsilon) \prod_{i=1}^{N/2} [\omega - (N - 2i + 1)]^2 \end{aligned} \right\} ; N \text{ even} \quad (3.4)$$

$$\left. \begin{aligned} Q_+ &= (a_{2N} - 1 - \epsilon) (\omega - N) \prod_{i=1}^{\frac{N-1}{2}} [\omega - (N - 2i)]^2 \\ Q_- &= (a_{2N} - 1 + \epsilon) \omega \prod_{i=1}^{\frac{N-1}{2}} [\omega - (N - 2i + 1)]^2 \end{aligned} \right\} ; N \text{ odd.}$$

The constant a_{2N} can be used to satisfy requirement (d); then using (2.13) with $\epsilon = 1$ the final result

$$R_{eH_N}(\omega^2) = \frac{4 \prod_{i=1}^{N/2} [\omega^2 - (N - 2i + 1)^2]^2}{2 \prod_{i=1}^{N/2} [\omega^2 - (N - 2i + 1)^2]^2 - \omega^2 (\omega^2 - N^2) \prod_{i=1}^{\frac{N-2}{2}} [\omega^2 - (N - 2i)^2]^2} ; N \text{ even} \quad (3.5)$$

$$R_{eH_N}(w^2) = \frac{4w^2 \prod_{i=1}^{\frac{N-1}{2}} [w^2 - (N - 2i + 1)^2]^2}{2w^2 \prod_{i=1}^{\frac{N-1}{2}} [w^2 - (N - 2i + 1)^2]^2 - (w^2 - N^2) \prod_{i=1}^{\frac{N-1}{2}} [w^2 - (N - 2i)^2]^2} ; N \text{ odd} \quad (3.5)$$

valid in $[-N, N]$ is obtained. With $N = 1, 2, 3$, and 4 , the above reduces to:

$$R_{eH_1}(w^2) = \frac{4w^2}{w^2 + 1}$$

$$R_{eH_2}(w^2) = \frac{4(w^2 - 1)^2}{w^4 + 2}$$

(3.6)

$$R_{eH_3}(w^2) = \frac{4w^2(w^2 - 4)^2}{w^6 - 5w^4 + 13w^2 + 9}$$

$$R_{eH_4}(w^2) = \frac{4(w^2 - 1)^2(w^2 - 9)^2}{w^8 - 16w^6 + 92w^4 - 104w^2 + 162}$$

By inspection, (3.6) satisfies the properties of the real part of a positive real driving point function.

Example (3.2)

A rational function, G_N , is to be obtained with the following constraints:

- a. approximates the degenerate polynomial "zero" in an equiripple manner
- b. $\lim_{w \rightarrow \infty} R_{eG_N} = 0$
- c. $R_{eG_N}(n^2) = (-1)^{2n}$ for $n = 0, \pm 1, \pm 2, \dots, \pm (N-1)$
- d. $R_{eG_N}\left(\frac{1}{2^2}\right) = 0$
- e. has $2N-1$ extreme points.

The rational function to be determined is sketched for positive frequencies in Figure 8.

Solution:

This problem appears to fit Case IV, however, two differences are noted; (1) the approximation interval extends from minus to plus infinity, and (2) the number of extreme points is $2N-1$ which is less than the maximum $2N+1$.

To modify Case IV such that it meets the stated requirements, imaginary zeros are placed at $u = \pm ja$ in Q_+ (Q_-) for N even (odd), thus giving:

$$\left. \begin{aligned} Q_+ &= -\omega^2 (\omega^2 + a^2) \prod_{i=1}^{\frac{N-2}{2}} (\omega^2 - 4i^2)^2 \\ Q_- &= \prod_{i=1}^{N/2} [\omega^2 - (2i-1)^2]^2 \end{aligned} \right\} ; N \text{ even}$$

$$\left. \begin{aligned} Q_+ &= -\omega^2 \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - 4i^2)^2 \\ Q_- &= (\omega^2 + a^2) \prod_{i=1}^{\frac{N-1}{2}} [\omega^2 - (2i-1)^2]^2 \end{aligned} \right\} ; N \text{ odd.}$$

(3.7)

Substituting these values in (2.13)

$$R_{eG_N} = \frac{\prod_{i=1}^{N/2} [\omega^2 - (2i-1)^2]^2 - \omega^2 (\omega^2 + a^2) \prod_{i=1}^{\frac{N-2}{2}} (\omega^2 - 4i^2)^2}{\prod_{i=1}^{N/2} [\omega^2 - (2i-1)^2]^2 + \omega^2 (\omega^2 + a^2) \prod_{i=1}^{\frac{N-2}{2}} (\omega^2 - 4i^2)^2} ; N \text{ even} \quad (3.8)$$

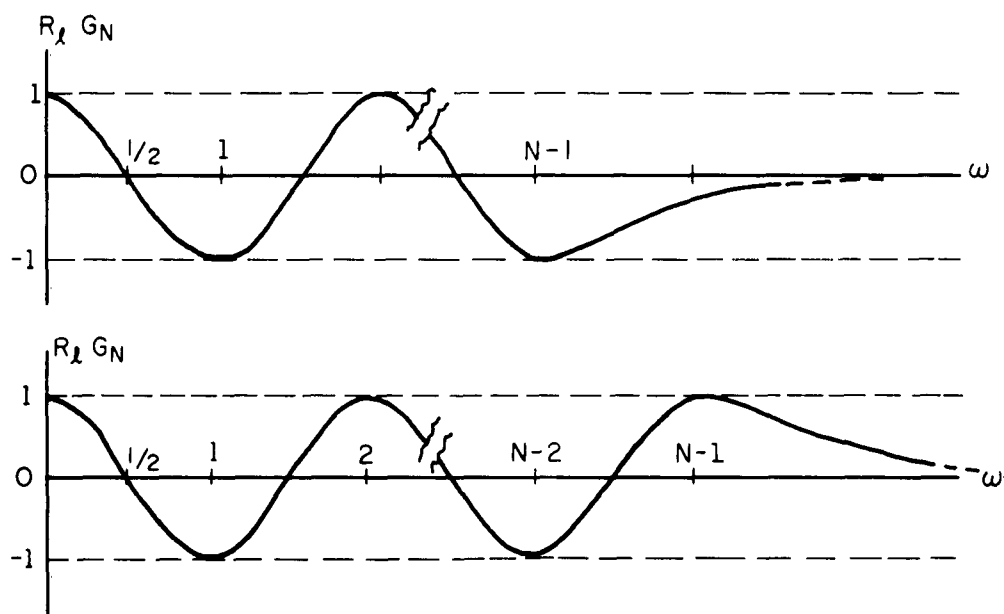


Figure 8. The real part of a network function which approaches zero for large frequencies

$$R_{eG_N} = \frac{(\omega^2 + a^2) \prod_{i=1}^{\frac{N-1}{2}} [\omega^2 - (2i-1)^2]^2 - \omega^2 \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - 4i^2)^2}{(\omega^2 + a^2) \prod_{i=1}^{\frac{N-1}{2}} [\omega^2 - (2i-1)^2]^2 + \omega^2 \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - 4i^2)^2} ; N \text{ odd.} \quad (3.8)$$

As a^2 is arbitrary it can be specified so that the numerator of (3.8) is zero at $\omega = 1/2$. After several algebraic steps, the value of $N(N-1)$ is obtained for a^2 .

Later in this chapter R_{eG_N} will be used for another example, therefore, it is tabulated for $N = 2$ through 6 as follows:

$$R_{eG_2} = \frac{-4\omega^2 + 1}{2\omega^4 + 1}$$

$$R_{eG_3} = \frac{12\omega^4 - 27\omega^2 + 6}{2\omega^6 - 4\omega^4 + 5\omega^2 + 6}$$

$$R_{eG_4} = \frac{-24\omega^6 + 198\omega^4 - 372\omega^2 + 81}{2\omega^8 - 16\omega^6 + 38\omega^4 + 12\omega^2 + 81} \quad (3.9)$$

$$R_{eG_5} = \frac{40\omega^8 - 810\omega^6 + 4740\omega^4 - 7615\omega^2 + 1620}{2\omega^{10} - 40\omega^8 + 246\omega^6 - 380\omega^4 + 577\omega^2 + 1620}$$

$$R_{eG_6} = \frac{-60\omega^{10} + 2415\omega^8 - 31860\omega^6 + 155535\omega^4 - 239430\omega^2 + 50625}{2\omega^{12} - 80\omega^{10} + 1071\omega^8 - 5300\omega^6 + 10127\omega^4 + 6330\omega^2 + 50625}.$$

It is apparent that the above functions satisfy the properties of a real part function; however, it is not a positive real one.

Example (3.3)

A rational function R_{eK_N} is to be found which has the properties of a real part function and yet approximates $\cos \pi \omega$ within the interval $(-1, 1)$.

Solution:

The function developed in Example (3.1) approximates $\cos \pi \omega$ in the required interval; however, a better approximation can be obtained if the rational function is zero not only at $\omega = \pm 1/2$, but also at $\omega = \pm 3/2$. To satisfy this last added condition requires another arbitrary constant. A rational function is thus chosen with the maximum number of extreme points all except the last of which are separated by a unit increment. The last extreme point and the value of K_N at infinity will then be used to place two zero crossings at $\omega = \pm 1/2$ and $\pm 3/2$. A sketch of the desired rational function $R_{e K_N}$ is shown in Figure 9.

From an inspection of Figure 9, Q_+ and Q_- can be easily determined. Placing these quantities in (2.13) gives the rational function

$$R_{e K_N} = \frac{(a_{2N+1}) \prod_{i=1}^{N/2} [\omega^2 - (2i-1)^2]^2 + (a_{2N-1}) \omega^2 (\omega^2 - b^2) \prod_{i=1}^{N/2} (\omega^2 - 4i^2)^2}{(a_{2N+1}) \prod_{i=1}^{N/2} [\omega^2 - (2i-1)^2]^2 - (a_{2N-1}) \omega^2 (\omega^2 - b^2) \prod_{i=1}^{N/2} (\omega^2 - 4i^2)^2} ; N \text{ even} \quad (3.10)$$

$$R_{e K_N} = \frac{(a_{2N+1}) (\omega^2 - b^2) \prod_{i=1}^{N-1} [\omega^2 - (2i-1)^2]^2 + (a_{2N-1}) \omega^2 \prod_{i=1}^{N-1} (\omega^2 - 4i^2)^2}{(a_{2N+1}) (\omega^2 - b^2) \prod_{i=1}^{N-1} [\omega^2 - (2i-1)^2]^2 - (a_{2N-1}) \omega^2 \prod_{i=1}^{N-1} (\omega^2 - 4i^2)^2} ; N \text{ odd.}$$

Letting $\omega = 1/2$ and setting the numerator equal to zero gives the following equation:

$$\frac{(a_{2N+1})}{(a_{2N-1})} + \frac{(1 - 4b^2)}{(2N-1)^2 (2N-3)^2} \prod_{i=1}^{N-2} \left(\frac{1 + 4i}{3 - 4i} \right)^2 = 0 ; N \text{ even} \quad (3.11)$$

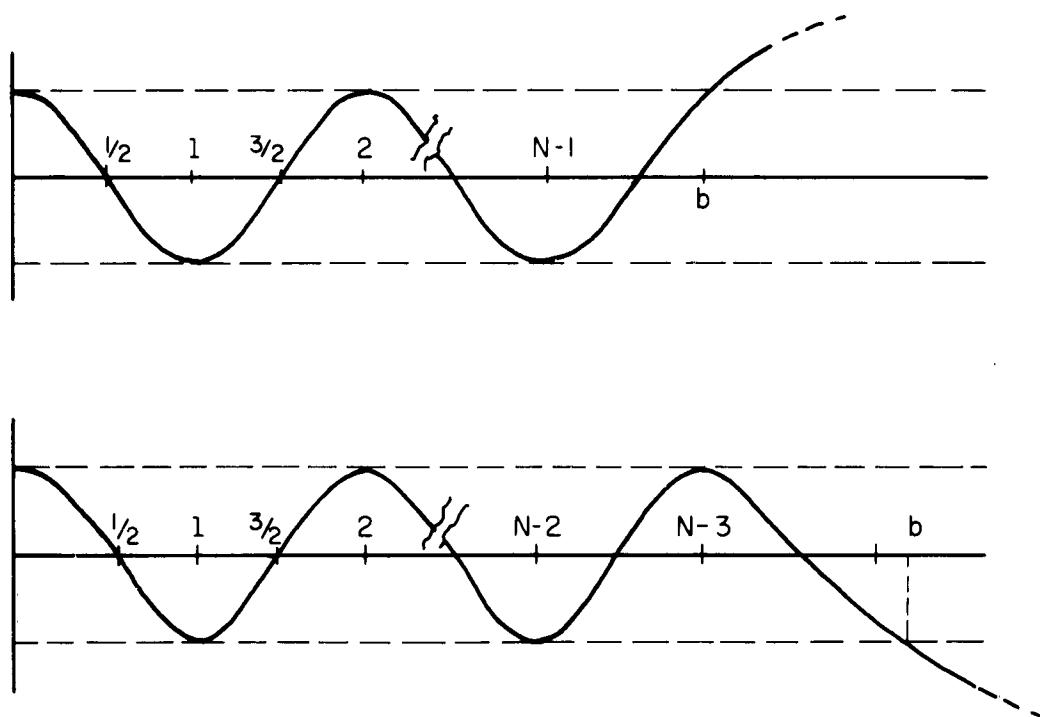


Figure 9. A rational function which approximates $\cos \pi \omega$

$$(1 - 4b^2) + \frac{(a_{2N} - 1)}{(a_{2N} + 1)} \prod_{i=1}^{\frac{N-1}{2}} \left(\frac{1 + 4i}{3 - 4i} \right)^2 = ; N \text{ odd.} \quad (3.11)$$

In a similar manner except with $\omega = 3/2$

$$\frac{(a_{2N} + 1)}{(a_{2N} - 1)} (2N + 1)^2 (2N - 5)^2 + 9(9 - 4b^2) \prod_{i=1}^{\frac{N-2}{2}} \left(\frac{3 - 4i}{1 + 4i} \right)^2 \left(\frac{3 + 4i}{5 - 4i} \right)^2 = 0 ; N \text{ even} \quad (3.12)$$

$$(9 - 4b^2) + 9 \frac{(a_{2N} - 1)}{(a_{2N} + 1)} \prod_{i=1}^{\frac{N-1}{2}} \left(\frac{3 - 4i}{5 + 4i} \right)^2 \left(\frac{3 + 4i}{5 - 4i} \right)^2 = 0 ; N \text{ odd} .$$

The appropriate values for b^2 and a_{2N} can now be obtained from the simultaneous solutions of the linear equations (3.11) and (3.12). After many algebraic steps the solutions

$$a_{2N} = (-1)^N 4N(N - 1) \quad (3.13)$$

$$b^2 = \frac{1}{4} \left[1 + \frac{(2N - 1)^4}{(2N - 1)^2 - 2} \right]$$

are obtained. It is now possible to determine the explicit expression for R_{eK_N} for any N . This is done for $N = 2$ through 6 as follows:

$$R_{eK_2} = \frac{16\omega^4 - 40\omega^2 + 9}{2\omega^4 + 4\omega^2 + 9}$$

$$R_{eK_3} = \frac{-48\omega^6 + 408\omega^4 - 747\omega^2 + 162}{2\omega^6 + 8\omega^4 + 53\omega^2 + 162} \quad (3.14)$$

$$R_{eK_4} = \frac{96\omega^8 - 1968\omega^6 + 11430\omega^4 - 18612\omega^2 + 3969}{2\omega^8 + 8\omega^6 + 134\omega^4 + 972\omega^2 + 3969}$$

(3.14)

$$R_{e K_5} = \frac{-160\omega^{10} + 6480\omega^8 - 85290\omega^6 + 417460\omega^4 - 636975\omega^2 + 134460}{2\omega^{10} + 246\omega^6 + 2740\omega^4 + 26577\omega^2 + 134460}$$

$$R_{e K_6} = \frac{240\omega^{12} - 16920\omega^{10} + 421335\omega^8 - 4501140\omega^6 + 19956375\omega^4 - 29216790\omega^2 + 6125625}{2\omega^{12} - 20\omega^{10} + 471\omega^8 + 4780\omega^6 + 88727\omega^4 + 1011690\omega^2 + 6125625}$$

A check of the accuracy of the sinusoidal approximation at $\omega = 1/4$ gives an error of .248, .0184 and .00296 percent for $N = 2, 3$ and 4 respectively.

This section on real part approximation will be completed after a novel technique is presented for obtaining linear phase all-pole transfer functions. From the theory of the Hilbert transform, if the real part of a network function is cosinusoidal then the imaginary part will be sinusoidal.¹³ Hence, the phase

$$\theta = \tan^{-1} \frac{\text{im part}}{\text{real part}} = \tan^{-1} \frac{\sin \pi \omega}{\cos \pi \omega} = \pi \omega \quad (3.15)$$

will be linear.

The one drawback to the above is that the sine is the Hilbert transform of the cosine only if the latter is defined for all frequencies. If, however, the real part is cosinusoidal over the finite frequency interval $(-\omega_0, \omega_0)$ and zero elsewhere, then the imaginary part will be roughly sinusoidal in the vicinity of $\omega = 0$. The reasoning for this last statement is best explained by noting the Hilbert transform of the cosine

$$\text{Im part} = + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos \xi \, d\xi}{\xi - \omega} \quad (3.16)$$

This integral relationship indicates that the behavior of the imaginary part, at a frequency ω_0 , is due primarily to the integrand within the vicinity of ω_0 . The modulation of the cosine function by $\frac{1}{\xi - \omega}$ is so slow outside this vicinity that the positive and negative areas nearly nullify each other.

The real part function developed in Example (3.2) is near cosinusoidal in the interval $(-N, N)$ and approaches zero elsewhere. Hence, using the well known Brune-Gewertz technique,³⁰ the function $G_N(s)$ can be obtained from $R_{e G_N}(\omega^2)$. This method involves the factorization of the denominator polynomial

and the solution of several simultaneous linear equations; these tasks make it a very laborious problem for high order real parts. To overcome these difficulties, a digital computer program was written to perform the necessary operations. The resulting $G_N(s)$ obtained by this program are given below for $N = 2-6$.

$$\begin{aligned}
 G_2 &= \frac{-1.08719s + .707107}{s^2 + 1.18721s + .707107} \\
 G_3 &= \frac{1.87170s^2 - 2.16633s + 1.73205}{s^3 + 2.04823s^2 + 3.09762s + 1.73205} \\
 G_4 &= \frac{-2.64461s^3 + 4.34677s^2 - 10.1194s + 6.36396}{s^4 + 2.89390s^3 + 8.18732s^2 + 10.4980s + 6.36396} \quad (3.17) \\
 G_5 &= \frac{3.41295s^4 - 7.25399s^3 + 33.0873s^2 - 39.2985s + 28.4605}{s^5 + 3.73461s^4 + 16.9736s^3 + 34.8719s^2 + 47.6806s + 28.4605} \\
 G_6 &= \frac{-4.17925s^5 + 10.8879s^4 - 81.6213s^3 + 139.974s^2 - 252.955s + 159.099}{s^6 + 4.57310s^5 + 30.4566s^4 + 86.7753s^3 + 200.779s^2 + 258.945s + 159.099}
 \end{aligned}$$

Though it isn't immediately apparent without factoring the numerator, all the zeros of $G_N(s)$ are in the right half of the s -plane. Furthermore, the phase of a right-half plane zero at $\sigma_1 + j\omega_1$ is indistinguishable from the phase of a left half plane pole at $-\sigma_1 + j\omega_1$. Therefore, if an all pole transfer function $\bar{G}_N(s)$ is desired, it can be obtained from $G_N(s) = N(s)/D(s)$ by replacing s with $-s$ in the numerator $N(s)$, and then placing N in the denominator. Hence,

$$\bar{G}_N(s) = \frac{A}{N(-s) D(s)} \quad \text{where } A \text{ is a constant.}$$

From the $\bar{G}_N(s)$ so obtained, the group delay response was calculated and plotted as shown in Figure 10. From this plot it is concluded that:

- a. The group delay approximates π as predicted in (3.15).
- b. The approximation is better near the origin as predicted.
- c. The approximation is better near the origin for higher values of N .

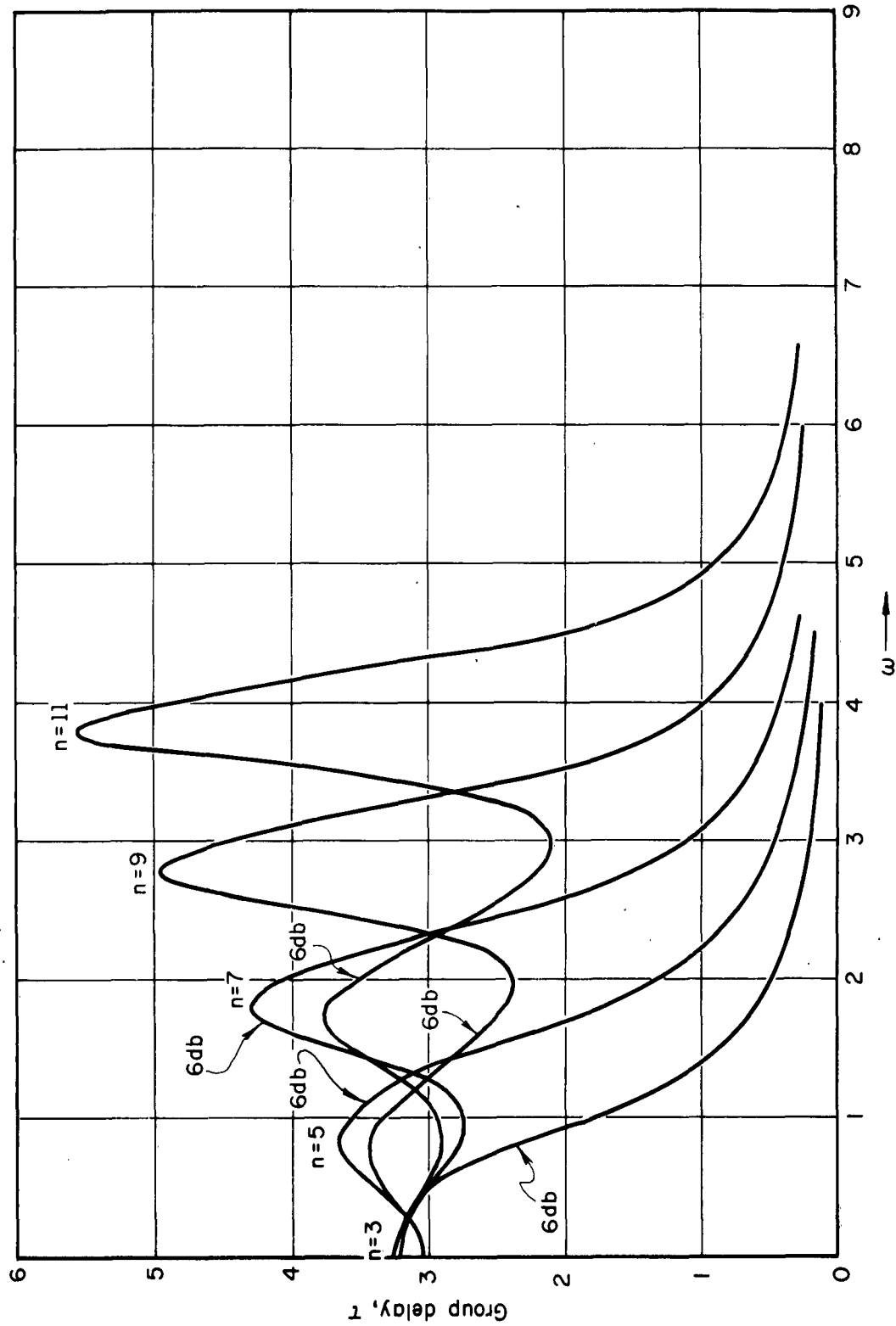


Figure 10. Group delay of an all pole transfer function which has been obtained by real port synthesis. n is the number of poles

- d. The group delay is relatively constant within the 6 db passband.
- e. The number of group delay ripples is not a maximum.

3.2 Imaginary Part Approximation

The imaginary part of a network function is an odd rational function of the real variable ω . This is easily shown by considering the odd part

$$\text{Odd } H(s) = \frac{N_1 M_2 - N_2 M_1}{M_2^2 - N_2^2} \quad (3.18)$$

of the network function of (3.1). It is now clear that upon letting $s = j\omega$ the above will be an odd rational function in the real variable ω multiplied by j . Hence, it is the imaginary part.

The primary property of the imaginary part is that it is odd. However, if the rational function of Chapter 2 is to be identified as an imaginary part, it must also possess the properties³⁰

- a. the numerator polynomial must be odd
- b. the poles must have quadrantal symmetry
- c. the degree of the numerator must be less than or equal to the degree of the denominator plus one, equality holding if $H(s)$ has a pole at $s = \infty$.

Several specific examples will now be given wherein an odd polynomial is approximated by an odd rational function to give an equiripple error. In addition it will be seen that the rational function so obtained also satisfies the above listed properties of an imaginary part.

Example (3.4)

An odd rational function is to be found which

- a. approximates the polynomial $P_1 = \omega$ in the interval $(-10, 10)$.
- b. has a peak error of unity in the passband
- c. has a 4th degree denominator and 1st degree numerator, and
- d. satisfies the properties of an imaginary part of a network function.

Such a rational function is illustrated in Figure 11.

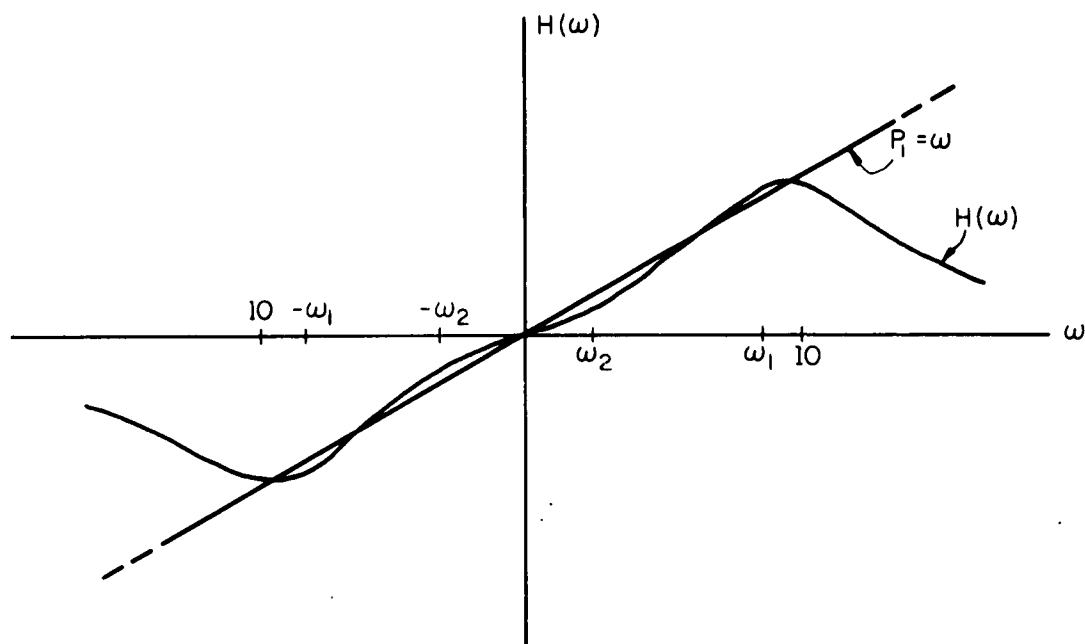


Figure 11. An odd rational function approximating the polynomial $P_1 = \omega$

Solution:

Both the rational function and the polynomial are odd, hence the rational function can be determined from the simplified form of (2.20). By inspection:

$$H(\omega) = \frac{\text{Odd } (\omega - 1) (\omega + 10) (\omega + \omega_2)^2 (\omega - \omega_1)^2}{\text{Ev } (\omega + 10) (\omega + \omega_2)^2 (\omega - \omega_1)^2} \quad (3.19)$$

Obviously, the denominator is fourth degree, but the numerator is of fifth degree. Thus, ω_1 and ω_2 will have to be used to set the coefficients of ω^5 and ω^3 equal to zero.

The coefficient of ω^5 is immediately observed to be

$$a_5 = 9 - 2\omega_1 + 2\omega_2 = 0. \quad (3.20)$$

Solving for ω_1

$$\omega_1 = \omega_2 + \frac{9}{2}. \quad (3.21)$$

With this relationship (3.19) can be manipulated to give

$$H(\omega) = \frac{(-9\omega_2^2 - \frac{81}{2}\omega_2 + \frac{1089}{4})\omega^3 + (9\omega_2^4 + 81\omega_2^3 + \frac{369}{4}\omega_2^2 - 405\omega_2)\omega}{(\omega^4 - 11\omega_2^2 + \frac{99}{2}\omega_2 - \frac{405}{2})\omega^2 + 10(\omega_2^4 + 9\omega_2^3 + \frac{81}{4}\omega_2^2)} \quad (3.22)$$

Setting

$$-9\omega_2^2 - \frac{81}{2}\omega_2 + \frac{1089}{4} = 0 \quad (3.23)$$

satisfies all requirements of the stated problem. The values of ω_2 and ω_1 , determined from (3.21) and (3.23) are

$$\begin{aligned} \omega_2 &= 3.69243 \\ \omega_1 &= 8.19243. \end{aligned} \quad (3.24)$$

Thus, the final form of H is

$$H(\omega) = \frac{5513.06\omega}{\omega^4 - 130.250\omega^2 + 9150.62} \quad (3.25)$$

This function satisfies the properties of an imaginary part, and a $Z(s)$ could be found with the above $H(\omega)$ as its imaginary part if desired.

Example (3.5)

Whenever high order polynomials are to be approximated by rational functions whose numerator degree n is less than the degree of the denominator m , several complicated, non-linear, simultaneous equations have to be solved. This example is chosen to illustrate this point. In the interval $(-10, 10)$ an odd rational function G is to be obtained such that it approximates $P_3 = \omega^3$ in an equiripple manner. Two solutions are obtained. One gives a maximum error ripple of one with the degrees m and n equal to 4 and 3 respectively. The other solution considers the problem with m and n equal to 4 and 1 respectively. The anticipated approximation is shown in Figure 12.

Solution:

By inspection of Figure 12 and using (2.20)

$$G(\omega) = \frac{\text{Odd } (\omega^3 - \epsilon) (\omega + 10) (\omega + \omega_3)^2 (\omega - \omega_4)^2 (\omega - \omega_2)^2}{\text{Ev } (\omega + 10) (\omega + \omega_3)^2 (\omega - \omega_4)^2 (\omega - \omega_2)^2} \quad (3.26)$$

Note that as (3.26) now stands, $m = 6$ and $n = 9$. It appears as though four coefficients have to be set equal to zero in order to satisfy the first sub-case. Upon closer examination of (3.26) however, it is seen that the coefficients of ω^6 and ω^9 are identical, thus reducing the number of simultaneous equations to three.

The polynomial coefficients of the numerator, $N(\omega)$ and denominator, $D(\omega)$, are obtained in a straightforward, but lengthy, manner. They are

$$\begin{aligned} N(\omega) = & ((10+2a)\omega^9 + (10a^2+20b+2c+2ab-\epsilon)\omega^7 + [10b^2+20ac+2bc-\epsilon(20a+a^2+ab)]\omega^5 \\ & + [10c^2-\epsilon(20c+20ab+b^2+2ac)]\omega^4 - \epsilon(20bc+c^2)\omega \end{aligned} \quad (3.27)$$

$$D(\omega) = (10+2a)\omega^6 + (10a^2+20b+2c+2ab)\omega^4 + (10b^2+20ac+2bc)\omega^2 + 10c^2$$

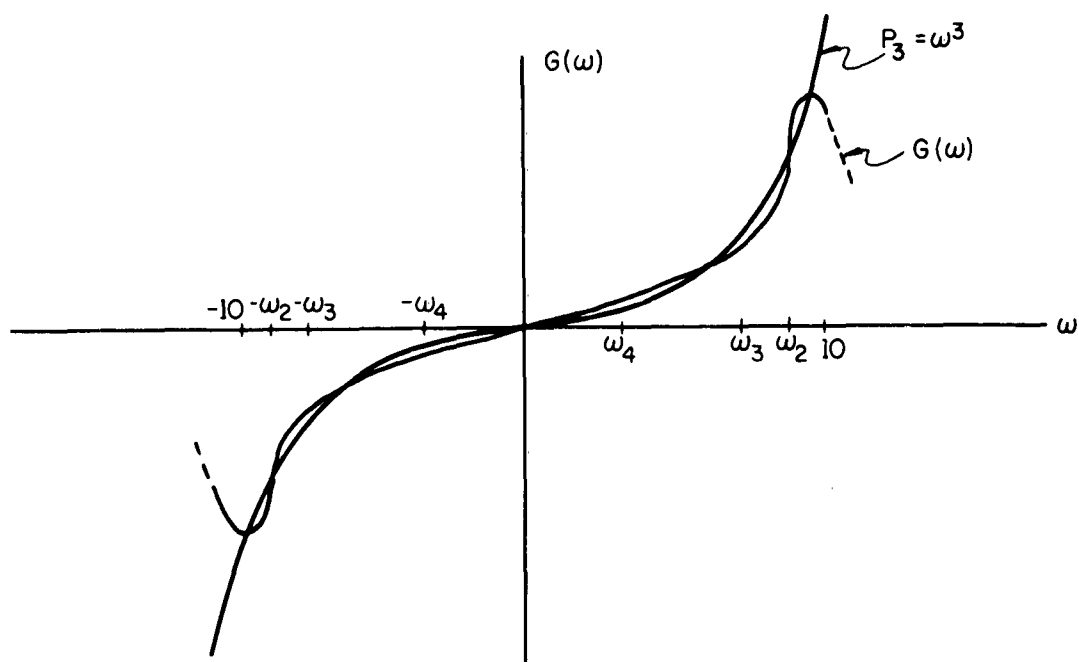


Figure 12, A rational function approximation of ω^3

where

$$\begin{aligned} a &= -\omega_2 + \omega_3 - \omega_4 \\ b &= -\omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4 \\ c &= \omega_2\omega_3\omega_4 \end{aligned} \quad (3.28)$$

Setting the coefficients of ω^9 , ω^7 , and ω^5 equal to zero and solving the resulting equations for a, b, and c in terms of ϵ , gives

$$\begin{aligned} a &= -5 \\ b &= -50 \frac{250 + \frac{\epsilon}{2}}{250 - \epsilon} \\ c &= 125 \frac{125 + \frac{3}{2}\epsilon - \frac{\epsilon^2}{500}}{125 - \frac{\epsilon}{2}} \end{aligned} \quad (3.29)$$

It is noted that if ϵ is known $G(\omega)$ can be obtained directly from a, b, and c without having a knowledge of ω_2 , ω_3 , and ω_4 . If, however, these values are desired, the cubic

$$B^3 + aB^2 + bB + c = 0 \quad (3.30)$$

can be factored, the zeros of which are then ω_2 , ω_4 , and $-\omega_3$.

Letting $\epsilon = 1$, the equations (3.29) can be used to obtain a, b, and c. From these and (3.27), the coefficients of the ω terms can be calculated. The G so obtained is

$$G(\omega) = \frac{152474.91\omega^3 + 111640.59\omega}{\omega^4 - 175.6024\omega^2 + 161305.3} \quad (3.31)$$

The frequencies of maximum error can be obtained from (3.30). They are

$$\begin{aligned} \omega_4 &= 2.24838 \\ \omega_3 &= 6.26492 \\ \omega_2 &= 9.01654 \end{aligned} \quad (3.32)$$

This is one of the required solutions to the approximation problem. The other is more difficult to obtain in that the ripple factor ϵ must be used

to set the coefficient of ω^3 equal to zero. This particular coefficient is a function of a , b , c , and ϵ . Since, however, the dependence of a , b , and c upon ϵ is known from (3.29), this coefficient can be determined solely as a function of ϵ . With this coefficient set equal to zero, and after the usual algebra, the following fourth order equation in ϵ results:

$$1 - 8 \left(\frac{\epsilon}{250} \right) + 3 \left(\frac{\epsilon}{250} \right)^2 + 5 \left(\frac{\epsilon}{250} \right)^3 - \left(\frac{\epsilon}{250} \right)^4 = 0 \quad (3.33)$$

Factoring the above, the possible ϵ 's are found to be

$$\epsilon = 33.26862; 250; 1321.9980; -355.26665 \quad .$$

From these roots it is seen that even the smallest value of ripple is comparatively large. Using the smallest value the corresponding G is

$$G(\omega) = \frac{205383.5\omega}{\omega^4 - 198.02523\omega^2 + 11927.04} \quad (3.34)$$

The maximum error then occurs at the frequencies

$$\begin{aligned} \omega_4 &= 2.94838 \\ \omega_3 &= 7.25754 \\ \omega_2 &= 9.30916 \quad . \end{aligned} \quad (3.35)$$

If desired, the Brune-Gewertz method could be used to obtain a function $Z(s)$ which has either of the above $G(\omega)$ for its odd part.

3.3 Magnitude Approximation

The properties of the magnitude function are easily obtained. From the network function $H(s)$ defined in Section 3.1, the magnitude squared function is found to be:

$$|H(\omega)|^2 = H(j\omega) H(-j\omega) = \frac{M_1^2 - N_1^2}{M_2^2 - N_2^2} \bigg|_{s=j\omega} \quad (3.36)$$

From this it is concluded that any even rational function of ω having the properties that¹⁴

- a. all poles and zeros have quadrantal symmetry
- b. any pole or zero on the real frequency axis is of even multiplicity
- c. the function is positive for all frequencies

can be used for a magnitude squared function.

The magnitude squared function most often used for filters is conveniently expressed as⁶

$$|H(\omega^2)|^2 = \frac{1}{1 + \sigma^2 \omega^{2k} F(\omega^2)^2} ; k = 0, 1 \quad (3.37)$$

where σ is a positive real constant less than one and the function $\omega^{2k} F(\omega^2)$ is chosen in such a manner that it is small (usually of unity peak value) in the passband and large in the stopband. This $\omega^{2k} F(\omega^2)$ will then give a magnitude squared function which approximates the ideal square magnitude function.¹⁴

In Example 3.6 an $\omega^{2k} F(\omega^2)$ is found by the method of Chapter 2 such that the ideal low pass magnitude function, defined as²³

$$H_I = \begin{cases} 1 & \text{for } -1 \leq \omega \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

is approximated in an equiripple manner.

Example 3.6

Several $\omega^{2k} F(\omega^2)^2$ are to be found such that the degenerate polynomial $P_0 = 1/2$ is approximated with a ripple factor $\epsilon = 1/2$ in the interval $(-1, 1)$. Outside of this interval the approximating function is to be large.

Solution:

To satisfy the last statement of the problem demands that $n \geq m$. Hence, depending upon whether the equal or greater than sign is used, the approximation fits either Case II or III. For the present discussion assume the maximum number of ripples; then a desirable rational function would appear as shown in Figure 13. Using the notation of Figure 13, the rational function, call it $f_N(\omega^2)$ for brevity, is

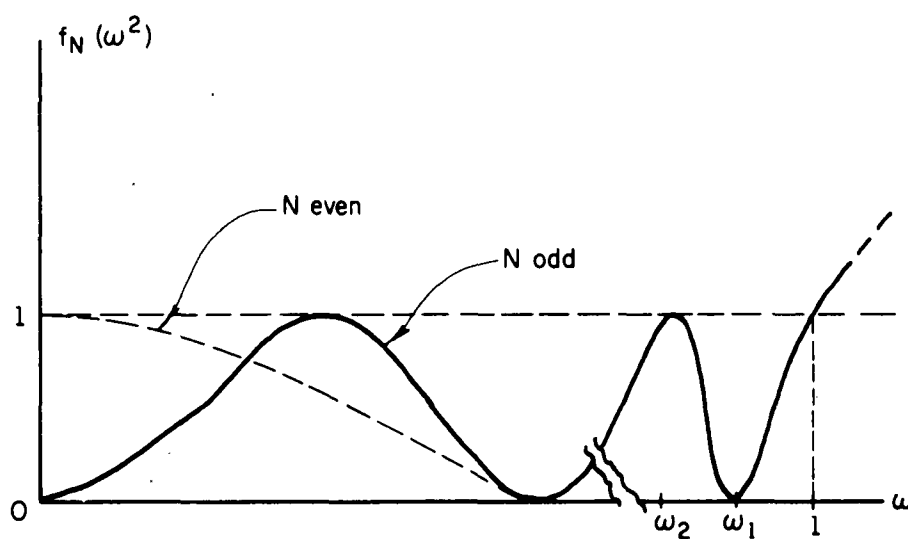


Figure 13, An equiripple rational function which can be used for filter magnitude approximations

$$f_N(\omega^2) = \frac{\omega^2 \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i-1}^2)^2}{\prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i-1}^2)^2 - \frac{a_{2N}^{-b}}{a_{2N}} (\omega^2 - 1) \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i}^2)^2} ; N \text{ odd} \quad (3.38)$$

$$f_N(\omega^2) = \frac{\prod_{i=1}^{\frac{N}{2}} (\omega^2 - \omega_{2i-1}^2)^2}{\prod_{i=1}^{\frac{N}{2}} (\omega^2 - \omega_{2i-1}^2)^2 - \frac{a_{2N}^{-b}}{a_{2N}} \omega^2 (\omega^2 - 1) \prod_{i=1}^{\frac{N-2}{2}} (\omega^2 - \omega_{2i}^2)^2} ; N \text{ even}$$

where

$$b = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n \end{cases}.$$

To obtain the classic Chebyshev polynomials the ω_i 's are so chosen that all the poles of (3.38) occur at $\omega = \text{infinity}$. This means that $b = 0$ and the $N-1$ denominator coefficients of ω^{2i} , $i=1, \dots, N-1$, are equated to zero. This gives $N-1$ equations and as many unknowns. Hence all ω_i 's are uniquely specified by the process. An illustration of the method for $N=4$ is given as follows: Expanding the denominator, the polynomial

$$\begin{aligned} &\omega^6 (1 + 2\omega_2^2 - 2\omega_1^2 - 2\omega_3^2) + \omega^4 (\omega_1^4 + 4\omega_3^4 + 4\omega_1^2 \omega_3^2 - 2\omega_2^2 - \omega_2^2) \\ &+ \omega^2 (\omega_2^2 - 2\omega_1^4 \omega_3^2 - 2\omega_3^4 \omega_1^2) + \omega_1^4 \omega_3^4 \end{aligned} \quad (3.39)$$

results. The 3 simultaneous equations are then

$$1 + 2\omega_2^2 = 2(\omega_1^2 + \omega_3^2) \quad (3.40)$$

$$2\omega_2^2 + \omega_2^4 = (\omega_1^2 + \omega_3^2)^2 + 2\omega_1^2 \omega_3^2$$

$$\omega_2^4 = 2\omega_1^2 \omega_3^2 (\omega_1^2 + \omega_3^2)$$

Letting $x = \omega_1^2 + \omega_3^2$; $y = \omega_1^2 \omega_3^2$ the solutions $x = 1$, and $y = 1/8$ are obtained. Using these results

$$f_4(\omega^2) = (8\omega^4 - 8\omega^2 + 1)^2$$

which is recognized as the square of the Chebyshev polynomial of 4th degree.⁸

Another classical design is the elliptic filter with an equiripple magnitude in both the pass and the stopbands.⁶ It can be shown that for such a filter the $f_N(\omega^2)$ function has poles and zeros which are reciprocally related, i.e., if ω_c is the proportionality constant,

$$\omega_{pi} \omega_{zi} = \omega_c \quad (3.41)$$

where ω_{pi} (ω_{zi}) is the location of the i th pole (zero).²¹ A sketch of $f_N(\omega^2)$ is shown in Figure 14.

The elliptic functions for $N=2$ and 3 can be obtained rather easily from (3.38). Letting $b = 1$ and 0 for these two respective cases, the required polynomial equivalences are

$$(\omega^2 - \omega_1^2)^2 - \left(\frac{a_4 - 1}{a_4}\right) \omega^2 (\omega^2 - 1) = \frac{1}{M^2} \left(\omega^2 - \frac{\omega_c^2}{\omega_1^2}\right)^2 ; N = 2 \quad (3.42)$$

$$\omega^2 (\omega^2 - \omega_1^2)^2 - (\omega^2 - 1) (\omega^2 - \omega_2^2)^2 = \frac{1}{M^2} \left(\omega^2 - \frac{\omega_c^2}{\omega_1^2}\right)^2 ; N = 3 .$$

Equating coefficients and solving for the unknowns in terms of ω_c and \bar{M} respectively, gives the following solutions

$$f_2(\omega^2) = M^2 \frac{(\omega^2 - \omega_1^2)^2}{(\omega^2 - \frac{\omega_c^2}{\omega_1^2})^2} \quad (3.43)$$

where

$$M = (\omega_c + \sqrt{\omega_c^2 - 1})^2$$

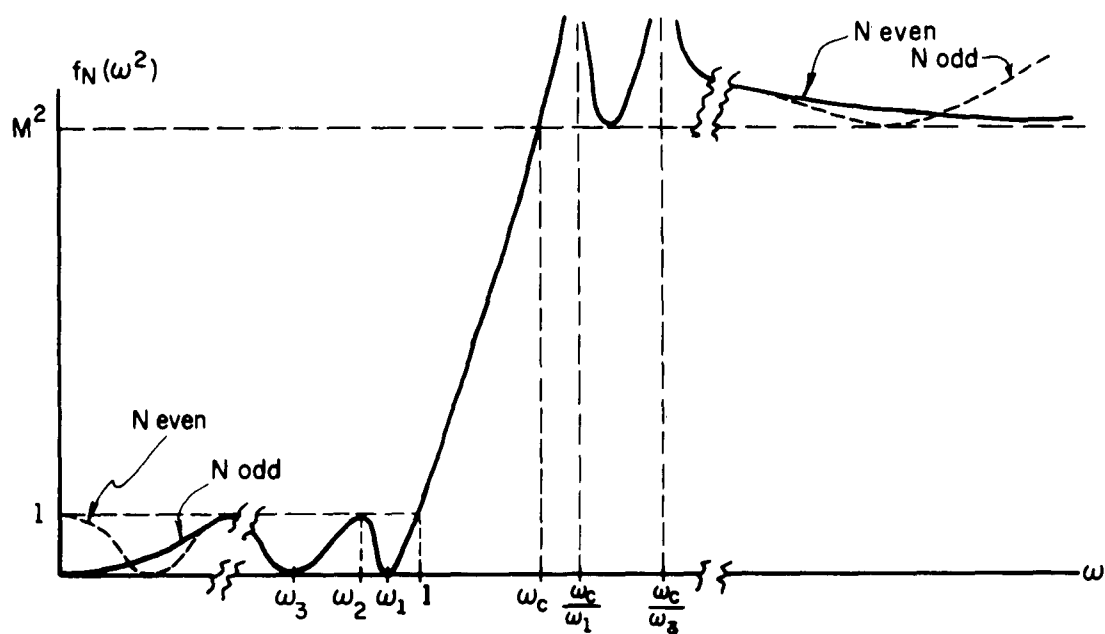


Figure 14. The f_N function for the elliptic filter

$$\omega_1^2 = \omega_c^2 (1 - \sqrt{1 - 1/\omega_c^2}).$$

Similarly,

$$f_3(\omega^2) = \bar{M}^2 \frac{\omega^2 (\omega^2 - \omega_1^2)^2}{(\omega^2 - \frac{\omega_c^2}{\omega_1^2})^2} \quad (3.44)$$

where

$$\omega_2^2 = \left(\frac{\bar{M} + 1}{\bar{M}^2} \right)^2$$

$$\omega_1^2 = \frac{3\bar{M} - 1}{\bar{M} + 1} \omega_2^2$$

$$\omega_c^2 = \bar{M} \omega_2^4$$

$$\bar{M}^2 = f_3(\omega_c^2).$$

Both solutions agree with those given by Papoulis²¹, who arrived at them in a somewhat different manner.

For higher orders of N , extensions of the above procedures would lead to several non-linear simultaneous equations; however, because this problem has been given a more elegant solution in the literature^{31,6} it will not be pursued further.

3.4 Constant Group Delay Approximation

The group delay, τ , of the all-pole transfer function

$$H(s) = \frac{A}{M_2 + N_2} = \frac{A}{N \prod_{i=1} (s - s_i)} ; \operatorname{Re} s_i < 0 \quad (3.45)$$

where $M_2 + N_2$ are the even and odd parts respectively of a Hurwitz polynomial, can be expressed in one of two ways. The first¹⁴

$$\tau = \frac{1}{j} \frac{M_2 N_2' - N_2 M_2'}{M_2^2 - N_2^2} \bigg|_{s=j\omega} \quad \text{where ' indicates differentiation} \quad (3.46)$$

is the polynomial representation. The second²⁹

$$\tau = \frac{1}{2} \sum_{i=1}^N \left[\frac{1}{s - s_i} - \frac{1}{s + s_i} \right] \bigg|_{s=j\omega} = \sum_{i=1}^N \frac{s_i}{s^2 - s_i^2} \bigg|_{s=j\omega}$$

is in terms of the left-half s-plane pole locations of H(s).

Using analytical continuation such that τ is valid for all complex s , the following properties of τ can be deduced:²⁹

- a. $\tau(s)$ has two zeros at $s = \infty$
- b. $\tau(s)$ is an even function
- c. $\tau(s)$ is positive, real and bounded for $s = j\omega$.

The necessary and sufficient conditions that an even rational function of s be a group delay function of an all pole transfer function are:²⁹

- a. The poles have quadrantal symmetry, and there are no poles on the $j\omega$ axis.
- b. The residues of left-half plane poles are $1/2$, right half plane poles $-1/2$. Equivalently if s^2 is replaced by p , the residue of any pole is the square root of the pole location.

From the foregoing, it is now possible to pick one of the equiripple rational functions developed in Chapter 2 and force it into being a group delay function. At the outset, it is noted that the rational function chosen must (1) be an even function and (2) have a numerator degree of two less than the denominator. Condition (2) and the fact that a constant is being approximated fix the approximation to Case IV. The degree of the denominator will be $2N$, hence, from Table 1 the total number of arbitrary constants is $n + 4 = 2N + 2$. Two of these constants are specified by fixing the polynomial $P_0 = c_0$ and the relative ripple ϵ . Also from Table 1, the maximum number of peak deviations is

$2N + 1$. Because the function is even, one extreme point will always occur at the origin; the others will be symmetrical about $\omega = 0$. Thus, the total maximum number of arbitrary peak deviations is N , but this is exactly the number of arbitrary constants yet to be specified. Therefore, it can be concluded that

- a. The polynomial $P_0 = c_0$ can be made any convenient value (for this problem let $c_0 = 1$).
- b. The ripple factor ϵ is arbitrary.
- c. The N pole locations can be fixed by varying the N unspecified peak deviation frequencies until the residue condition is met. The resulting left half plane poles of the equiripple rational function can then be used to obtain a transfer function $H(s)$ with an equiripple group delay.
- d. The equal ripple group delay will look similar to the illustration of Figure 15.

Using the notation of Figure 15 and the rational function in (2.14b), the proposed group delay, as a function of ω is:

$$\tau(\omega) = 1 + \epsilon - 2\epsilon \frac{\omega^2 \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i}^2)^2}{\prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i}^2)^2 - \frac{1-\epsilon}{1+\epsilon} (\omega^2 - \omega_1^2) \prod_{i=1}^{\frac{N-1}{2}} (\omega^2 - \omega_{2i}^2 + 1)^2}; \quad N \text{ odd} \quad (3.48)$$

$$\tau(\omega) = 1 + \epsilon - 2\epsilon \frac{\omega^2 \prod_{i=1}^{\frac{N}{2}} (\omega^2 - \omega_{2i}^2)^2}{\prod_{i=1}^{\frac{N}{2}} (\omega^2 - \omega_{2i}^2)^2 - \frac{1-\epsilon}{1+\epsilon} \omega^2 (\omega^2 - \omega_1^2) \prod_{i=1}^{\frac{N-2}{2}} (\omega^2 - \omega_{2i}^2 + 1)^2}; \quad N \text{ even.}$$

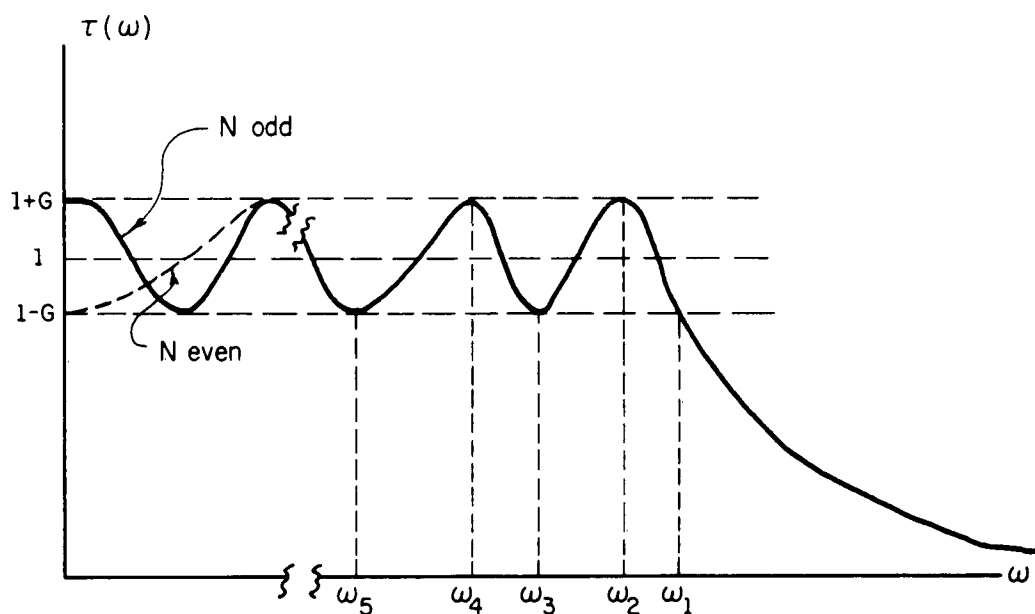


Figure 15. The equiripple group delay approximation of a constant

Using analytic continuation, i.e., letting $s^2 = -\omega^2$, and then, since τ is an even function, letting $s^2 = p$ the above becomes

$$\tau(p) = 1 + \epsilon - \frac{2 \in p \prod_{i=1}^{\frac{N-1}{2}} (p + u_{2i})^2}{p \prod_{i=1}^{\frac{N-1}{2}} (p + u_{2i})^2 - \frac{1-\epsilon}{1+\epsilon} (p + u_1) \prod_{i=1}^{\frac{N-1}{2}} (p + u_{2i+1})^2} ; N \text{ odd} \quad (3.49)$$

$$\tau(p) = 1 + \epsilon - \frac{2 \in \prod_{i=1}^{\frac{N}{2}} (p + u_{2i})^2}{\prod_{i=1}^{\frac{N}{2}} (p + u_{2i})^2 - \frac{1-\epsilon}{1+\epsilon} p (p + u_1) \prod_{i=1}^{\frac{N-2}{2}} (p + u_{2i+1})^2} ; N \text{ even}$$

where $u_i = \omega_i^2$; $i = 1, 2, \dots, N$.

It is interesting to note at this point that the results of Ulbrich and Piloty²⁹ could be duplicated by letting $u_1 = 1$ and letting the average delay be an unknown quantity. The ripple factor ϵ , under these circumstances, would be an absolute delay. Because of the transformation involved, Ulbrich and Piloty could not be this flexible.

Abele² did not use an explicit rational function in his work but imposed constraints on the numerator and denominator polynomials of (3.46). These constraints appear in a form similar to the numerator and denominator of (3.48).

Prior to the publication of Abele's work this author worked out closed form solutions to the equiripple group delay for $n = 1$ and 2 , using an approach similar to that of Abele. Helman¹⁶, using Bernstein's Chebyshev rational function, also obtained solutions for $n = 1$ and 2 . In Helman's technique, however, it is necessary to find the zeros of a high order polynomial to obtain the desired solution.

A closed form solution for the polynomial coefficients of the transfer functions

$$H_1 = \frac{a_o}{s + a_o} \quad (3.50)$$

$$H_2 = \frac{b_o}{s^2 + b_1 s + b_o}$$

such that the group delay is equiripple, will now be given. Using (3.46) the group delay of (3.50) is

$$\tau_1 = \frac{a_o}{a_o^2 - s^2} \quad (3.51)$$

$$\tau_2 = \frac{b_o b_1 - b_1 s^2}{b_o^2 + (2b_o - b_1^2) s^2 + s^4}$$

which, to be equiripple, must also be equal to (3.48). Using analytic continuation and simplifying, (3.48) becomes for $n = 1$ and 2

$$\tau_1 = \frac{\frac{(1 - \epsilon^2)}{2\epsilon} \omega_1^2}{s^2 - \frac{1 - \epsilon}{2\epsilon} \omega_o^2} \quad (3.52)$$

$$\tau_2 = \frac{\epsilon_2 (2\omega_2^2 - \omega_1^2) s^2 + \epsilon_2 \omega_2^4}{s^4 + (2\epsilon_4 \omega_2^2 - \epsilon_3 \omega_1^2) s^2 + \epsilon_4 \omega_2^4}$$

where

$$\epsilon_1 = \frac{1 - \epsilon}{1 + \epsilon}$$

$$\epsilon_2 = \frac{1 - \epsilon}{1 - \epsilon_1}$$

$$\epsilon_3 = \frac{\epsilon_1}{1 - \epsilon_1}$$

$$\epsilon_4 = \frac{1}{1 - \epsilon_1} \quad .$$

Equating the coefficients of (3.51) to those in (3.52) and solving the resulting simultaneous equations, the solutions for $n = 1$ and 2 are found to be

$$\begin{aligned} \underline{n = 1:} \quad \omega_1 &= \frac{1}{1 - \epsilon^2} \sqrt{2\epsilon(1 - \epsilon)} \\ a_0 &= \frac{1}{1 + \epsilon} \end{aligned} \quad (3.53)$$

$$\begin{aligned} \underline{n = 2:} \quad \omega_2^2 &= \frac{4\epsilon}{(1 - \epsilon^2)^2} \left[\frac{3 + \epsilon}{2} \sqrt{\frac{1 + \epsilon}{2\epsilon}} - (1 + \epsilon) \right] \\ \omega_1^2 &= \left(2 + \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right) \omega_2^2 \\ b_0 &= \sqrt{\frac{1 + \epsilon}{2\epsilon}} \omega_2^2 \\ b_1 &= \frac{1 - \epsilon^2}{2\epsilon} \sqrt{\frac{2\epsilon}{1 + \epsilon}} \omega_1^2 \end{aligned}$$

Obviously, the equations for n greater than two become so complicated that a closed form solution is no longer possible. Numerical answers, on the other hand, can still be obtained by using a high speed digital computer to perform the many operations required in an iterative solution.

There are many approaches which can be used in solving non-linear simultaneous equations.^{1,17} One method is the Newton-Raphson technique. This procedure has an advantage over many others in that, if the initial guess is within the region of convergence, the convergence is extremely rapid. A disadvantage of the method lies in the fact that the region of convergence for many problems of this type is small.¹ For the problem at hand the ω_1 's will be iterated. This means, if the Newton-Raphson procedure is used, that the initial guesses must be known reasonably accurately. Such an estimate can be made very easily if it is reasoned that the ω_1 's for increasing N will follow a definite pattern. Furthermore, this pattern has already begun with a closed form solution for $n = 1, 2$.

Briefly, the method of solution undertaken is as follows:

- a. From a knowledge of lower order solution, a set of ω_i 's is picked to correspond to a large ripple factor ϵ (it is found experimentally that the region of convergence is larger for large ϵ).
- b. Using these values of ω_i , the denominator of (3.49) is factored and the residues obtained.
- c. Knowing that the residues squared less the root locations must be zero gives N equations, F_i , to be satisfied in terms of the N unknowns ω_i , $i=1, \dots, N$.
- d. The partial derivatives $\partial F_i / \partial \omega_i$ are obtained numerically in the approximate form

$$\frac{\partial F_j}{\partial \omega_i} \approx \frac{F_j(\omega_i + \delta \omega_i) - F_j(\omega_i)}{\delta \omega_i}$$

where δ is a small constant.

- e. The N simultaneous equations (Newton-Raphson)

$$\sum_{i=1}^N \frac{\partial F_j}{\partial \omega_i} \Delta \omega_i = -F_j ; j = 1, \dots, N \quad (3.55)$$

are solved for the unknowns $\Delta \omega_i$.

- f. The initial values of ω_i are replaced by $\omega_i + \Delta \omega_i$.
- g. The above is repeated (usually from 4 to 8 times) until the ω_i 's are made as small as the numerical accuracy of the computer allows. The error commonly obtained is from one part in 10^{10} to one part in 10^7 for N ranging from 3 to 10.
- h. When the set of solutions ω_i is known, it is then used for the initial guess, and the ripple factor ϵ is given a smaller value.

The pole locations of the all pole equiripple group delay filter resulting from the above program are tabulated in Table 2 of the appendix for $N = 3, \dots, 10$, each with $\epsilon = 0.005, 0.01$, and 0.05 . For comparative purposes,

TABLE 2

Root locations of an N pole (N = 2 to 10) equiripple group delay filter for 3 values of ripple, ϵ . The normalization is such that the mean low frequency delay is unity.

	$\epsilon = 0.005$		$\epsilon = 0.01$		$\epsilon = 0.05$	
	real part	imag. part	real part	imag. part	real part	imag. part
2	-1.4021451	± 0.85237134	-1.3628732	± 0.94649670	-1.2039728	± 1.0416954
3	-1.9088240	0.0	-1.8061331	0.0	-1.4703023	0.0
	-1.5693900	± 2.0151812	-1.5022964	± 2.0802275	-1.2765428	± 2.2963311
4	-2.1050989	± 1.1051383	-1.9643081	± 1.1459939	-1.5460122	± 1.2621897
	-1.6498925	± 3.2352073	-1.5669472	± 3.3388487	-1.3079526	± 3.6530753
5	-2.3081926	0.0	-2.1262469	0.0	-1.6218304	0.0
	-2.1996486	± 2.3365664	-2.0381080	± 2.4144421	-1.5791674	± 2.6242135
	-1.6951436	± 4.5363797	-1.6028093	± 4.6721443	-1.3250626	± 5.0642095
6	-2.4057414	± 1.2339268	-2.2016824	± 1.2704024	-1.6550387	± 1.3628473
	-2.2532646	± 3.6461522	-2.0794685	± 3.7547734	-1.5974133	± 4.0389428
	-1.7235697	± 5.8905328	-1.6252314	± 6.0529615	-1.3357252	± 6.5084283
7	-2.5044409	0.0	-2.2777117	0.0	-1.6882400	0.0
	-2.4611483	± 2.5460921	-2.2440387	± 2.6127150	-1.6733428	± 2.7783913
	-2.2872444	± 5.0065049	-2.1055654	± 5.1407019	-1.6088716	± 5.4857180
	-1.7428862	± 7.2812168	-1.6404479	± 7.4661369	-1.3429725	± 7.9747231
8	-2.5604843	± 1.3128118	-2.3203970	± 1.3427945	-1.7065422	± 1.4157328
	-2.4963531	± 3.9086731	-2.2708293	± 4.0003415	-1.6848570	± 4.2258593
	-2.3105097	± 6.4018708	-2.1234089	± 6.5577090	-1.6167055	± 6.953096
	-1.7567883	± 8.6983493	-1.6513970	± 8.9025775	-1.3482052	± 9.4567365
9	-2.6168521	0.0	-2.3632467	0.0	-1.7248403	0.0
	-2.5961134	± 2.6761673	-2.3474107	± 2.7309954	-1.7180599	± 2.8634253
	-2.5205210	± 5.3058951	-2.2891886	± 5.4187559	-1.6927410	± 5.6946244
	-2.3273558	± 7.8226335	-2.1363241	± 7.9971129	-1.6223879	± 8.4373817
	-1.7672335	± 10.135398	-1.6596318	± 10.356477	-1.3521532	± 10.950485
10	-2.6526863	± 1.3635814	-2.3903646	± 1.3883671	-1.7363559	± 1.4477569
	-2.6205920	± 4.0741370	-2.3659368	± 4.1499642	-1.7259501	± 4.3324074
	-2.5380595	± 6.7281919	-2.3025079	± 6.8593217	-1.6984663	± 7.1785723
	-2.3400882	± 9.2625868	-2.1460838	± 9.4533690	-1.6266905	± 9.9322817
	-1.7753412	± 11.587925	-1.6660347	± 11.823921	-1.3552374	± 12.453322

Table 3 gives (a) the zero frequency delay (for a unity 6 db bandwidth), (b) the ratio of the 3 db to 6 db attenuation points, (c) the ratio of the equiripple approximation interval to the 6 db attenuation points, and (d) the percentage overshoot of the impulse response. Assuming the maximum of the equiripple group delay transfer function occurs at zero frequency (this assumption is true for N odd; and for N even if ϵ is smaller than about 0.05), the numerator constant of the transfer function can be made equal to the denominator constant and the magnitude characteristics vs. angular frequency can be computed. A plot of this characteristic is shown in Figure 16 for $N = 3, \dots, 10$ and $\epsilon = 0.005$. The corresponding attenuation curves for $0.005 \leq 0.05$ are but slightly different. As ϵ becomes larger (≈ 0.1) the magnitude response departs from that given in Figure 16. Two major differences are noted: (1) the individual poles in the transfer function become apparent by the lumpy nature of response, and (2) the high frequency attenuation is a few db greater.

For small ϵ , the attenuation curves are almost identical to those of the corresponding Bessel or maximally flat delay filter²⁸ up to about the 10 db attenuation points. For higher frequencies the attenuation becomes larger.

The applications of constant group delay filters are, primarily, for the transmission of pulsed information. Thus, the impulse response of such a filter is of prime interest. Two families of curves are given in Figures 17 and 18 for the impulse responses ($N = 3, \dots, 10$) with $\epsilon = 0.005$ and 0.05 respectively. From these responses the following conclusions can be drawn:

- a. The pulses are nearly symmetrical about the time $t = \tau_0$. This is particularly true for large N .
- b. The impulse response maximum is almost independent of N .
- c. The overshoot decreases with an increase in N .
- d. The curves are not plotted to sufficient accuracy to observe the peak of the over or under shoots. However, from the computer data it is noted that for $N > 7$ the over or undershoot approaches a constant of 0.2 and 2.0 percent for $\epsilon = 0.005$ and $\epsilon = 0.05$ respectively. The figures predicted in the introduction were about 0.1 and 1.0 percent respectively.

The reason the overshoot and asymmetry is greater for small N is that the group delay approximation interval is not as large as the passband of the

TABLE 3

With a unity 6 db radian frequency bandwidth (a) the low frequency mean delay, (b) the 3 db bandwidth, (c) the equiripple group delay approximation interval, and (d) the impulse response overshoot ($\epsilon = 0.005$, $\epsilon = 0.05$) are given.

$\epsilon = 0.005$

N	τ_o	$\omega_{3db}/\omega_{6db}$	ω_τ/ω_{6db}	$\delta\%$
2	1.97316	.701656	.389385	.
3	2.51335	.742110	.715918	2.3
4	3.11611	.731226	.962641	2.9
5	3.62909	.721310	1.18577	2.5
6	4.07041	.718634	1.38658	1.7
7	4.45741	.713878	1.57773	.9
8	4.82706	.717055	1.75023	.4
9	5.16313	.709500	1.91449	.2
10	5.48223	.716544	2.06791	.2

$\epsilon = 0.01$

2	1.97566	.707682	.460990	
3	2.53967	.744943	.793434	
4	3.16442	.727498	1.03260	
5	3.64538	.722188	1.26177	
6	4.11155	.717549	1.45456	
7	4.47961	.713319	1.65060	
8	4.86091	.718993	1.81672	
9	5.18980	.706297	1.98182	
10	5.50975	.720700	2.13315	

$\epsilon = 0.05$

2	2.01019	.737244	.668501	
3	2.63387	.742351	.992804	3.1
4	3.33354	.711244	1.19497	3.1
5	3.65745	.736920	1.47683	2.2
6	4.29086	.704986	1.59647	2.1
7	4.45445	.722572	1.86774	2.1
8	4.99602	.733237	1.96240	2.2
9	5.22731	.691956	2.16169	2.2
10	5.59256	.755409	2.28950	2.25

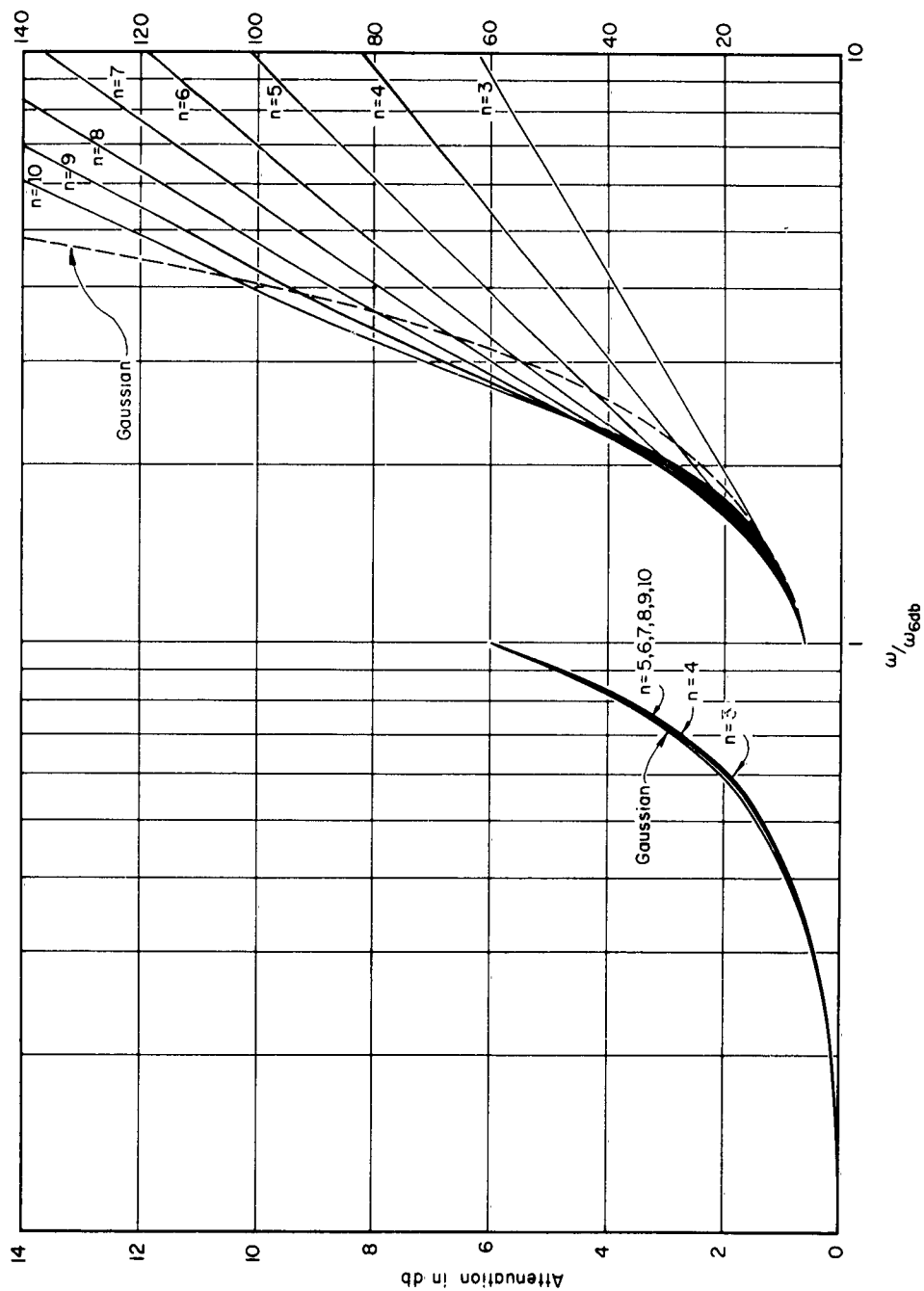


Figure 16. Attenuation of the Equiripple Group Delay Filter with a 1/2 percent ripple. The normalization is such that the 6 db bandwidth is unity (the ideal gaussian filter is shown for comparison)

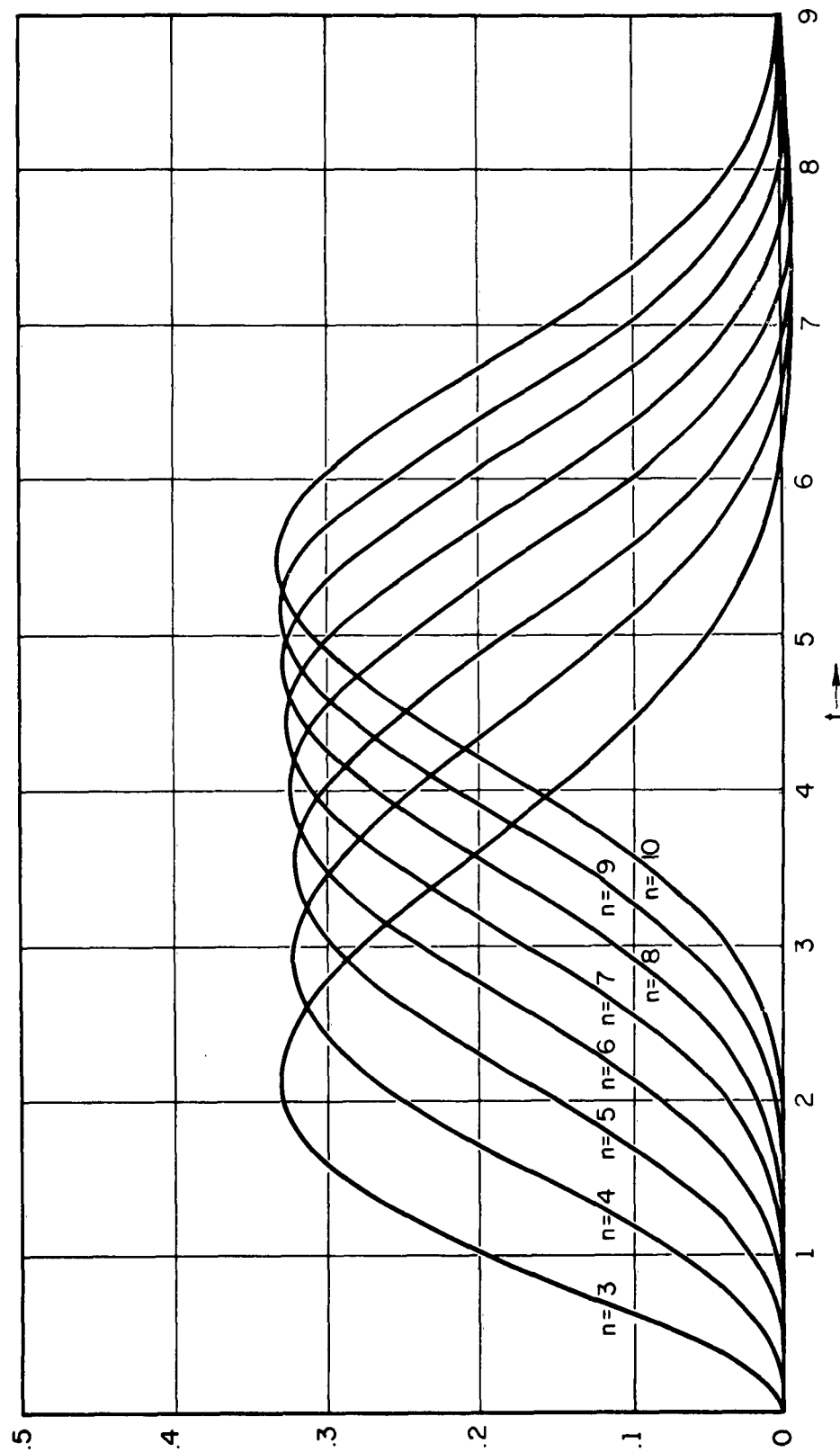


Figure 17. Impulse response of an equiripple group delay filter with a 6 db unity bandwidth and 1/2 percent ripple

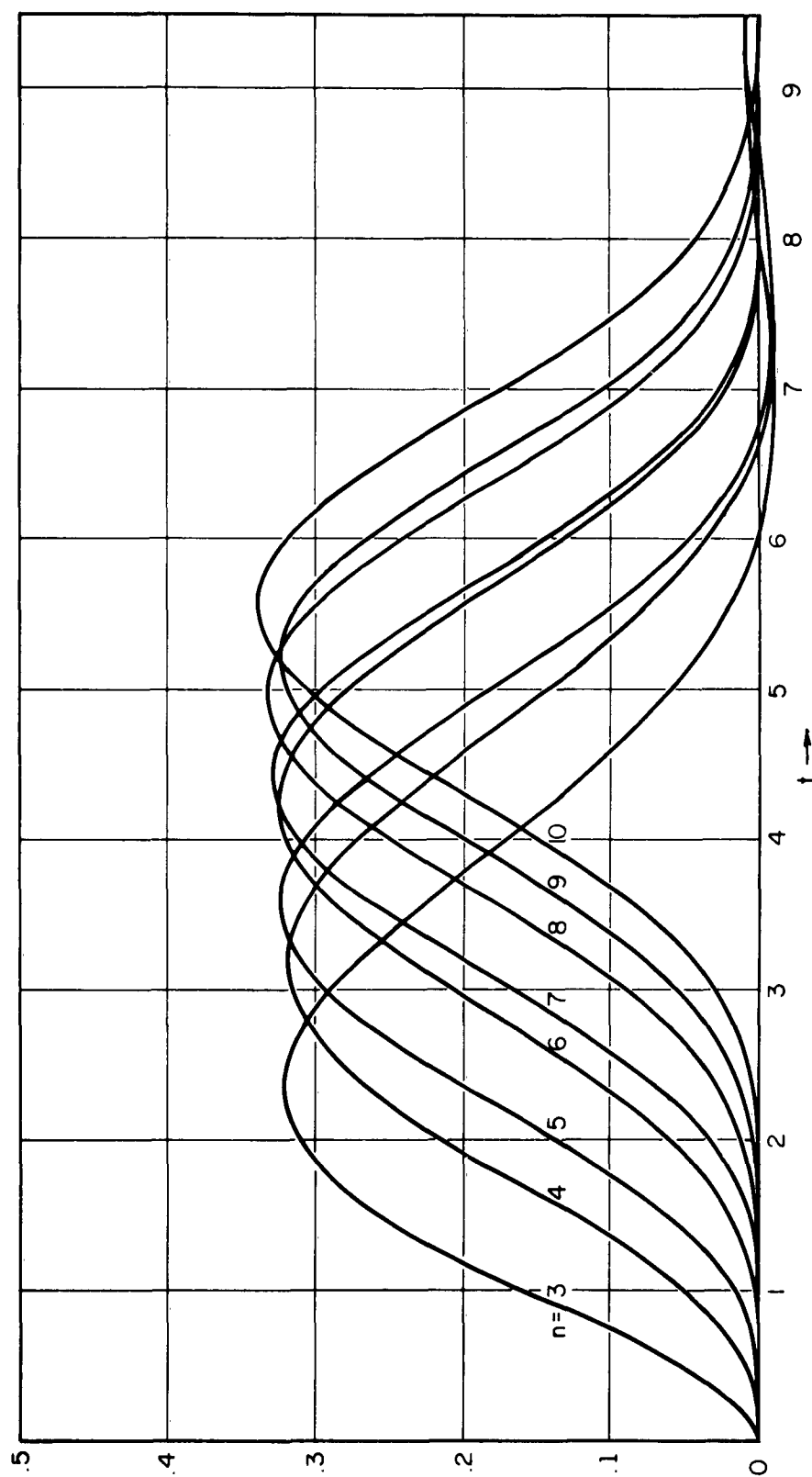


Figure 18. Impulse response of an equiripple group delay filter with a 6 db unity bandwidth and a 5 percent ripple

filter. Thus, the higher frequency components arrive out of phase causing irregularities in the impulse response.

3.5 Narrow Band Approximation of Linearly Decreasing Group Delay

This chapter will be concluded after the formulation of the equi-ripple, linearly decreasing, narrow band group delay problem has been investigated. A graphical solution to this problem has appeared in the literature²⁰, but as of yet an analytical one has not.

Consider an all pass transfer function

$$\tau(s) = \prod_{i=1}^N \frac{(s + s_i)(s + s_i^*)}{(s - s_i)(s - s_i^*)} \quad \text{where } \text{Res}_i < 0 \quad (3.56)$$

with an even number of complex poles. The numerator will give the same delay as the denominator. Therefore, the total delay can be obtained by inserting a multiplying factor of two into (3.47), giving

$$\tau(j\omega) = \sum_{i=1}^N \left[\frac{1}{s - s_i} + \frac{1}{s - s_i^*} - \frac{1}{s + s_i} - \frac{1}{s + s_i^*} \right] \bigg|_{s=j\omega} \quad (3.57)$$

Letting $s_i = -\sigma_i + j\omega_i$

$$\tau(j\omega) = \sum_{i=1}^N \left[\frac{1}{\sigma_i + j(\omega + \omega_i)} + \frac{1}{\sigma_i + j(\omega - \omega_i)} - \frac{1}{-\sigma_i + j(\omega + \omega_i)} - \frac{1}{-\sigma_i + j(\omega - \omega_i)} \right] \quad (3.58)$$

The narrow band approximation is obtained by noting that if $\omega_i \gg \sigma_i$ and $\omega_1 \approx \omega_2 \approx \dots \approx \omega_N$ then the first and third term of (3.58) are negligible for the frequency band of interest. With the definitions

$\omega_o = \text{a constant frequency displacement}$

$w = \omega - \omega_o = u + jv$

$u_i = \omega_i - \omega_o$

$v_i = \sigma_i$

the narrow band group delay is given in the approximate form

$$\begin{aligned}\tau(w) &= \sum_{i=1}^N \left[\frac{j}{w - u_i + jv_i} - \frac{j}{w - u_i - jv_i} \right] \\ &= \sum_{i=1}^N \left[\frac{j}{w - w_i^*} - \frac{j}{w - w_i} \right]\end{aligned}\quad (3.59)$$

where $w_i = u_i + jv_i$.

The following properties can now be observed about the narrow band group delay:

- a. There are two poles at $\omega = \infty$.
- b. The poles are symmetrical about the real axis.
- c. The residues are $\pm j$.
- c. The denominator is of even degree.
- e. The rational function is neither odd nor even.

From these properties it is evident that the above described rational function fits Case V. If such a function is to approximate the straight line $\tau_0 - ku$ in the interval $(0, u_1)$ with an equiripple error it will appear as shown in Figure 19. Reference to Table 1 indicates that the total number of arbitrary constants is $n - \ell + 5 = N + 2$. One of these constants ($u_{N+2} = 0$) has already been specified. It will require N degrees of freedom to satisfy the residue condition. This leaves one more constant which can be fixed independently. Let it be ϵ . From Figure 19 and (2.13) the equiripple narrow band group delay takes the form

$$\tau = -k \frac{(u - k_1)u \prod_{i=1}^N (u - u_{2i})^2 - (u - k_2)(u - u_1) \prod_{i=1}^N (u - u_{2i+1})^2}{u \prod_{i=1}^N (u - u_{2i})^2 - (u - u_1) \prod_{i=1}^N (u - u_{2i+1})^2} \quad (3.60)$$

where

$$\begin{aligned}k_1 &= \frac{\tau_0 + \epsilon}{k} \\ k_2 &= \frac{\tau_0 - \epsilon}{k}.\end{aligned}$$

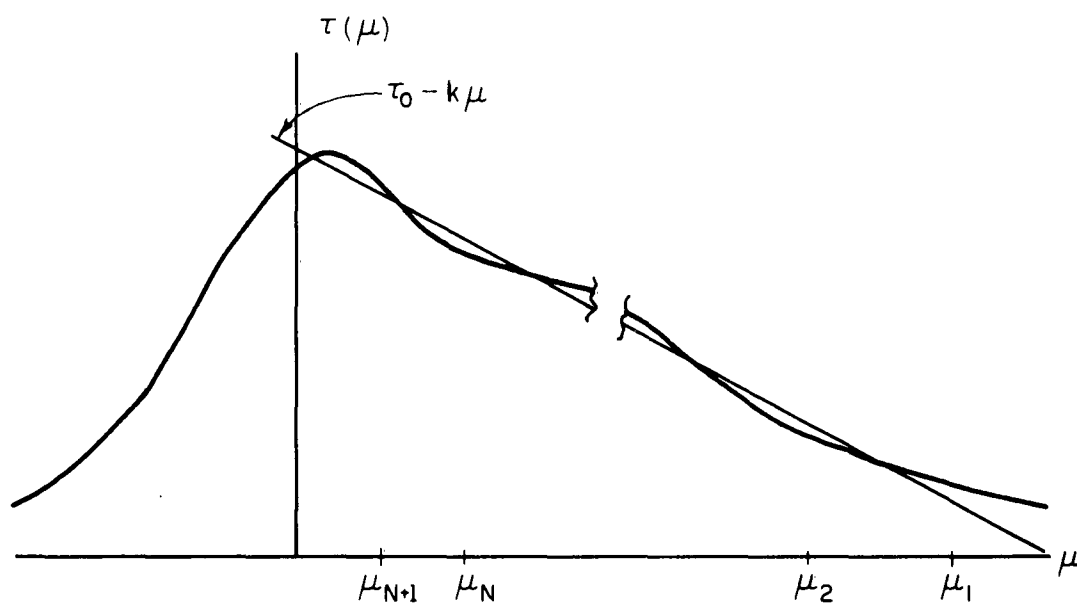


Figure 19. The narrow band group delay approximating a straight line in an equiripple manner

In order to make the numerator 2 degrees less than the denominator requires 3 simultaneous, non-linear equations. This in addition to the residue condition will fix the $N + 3$ unknown constants in (3.60). A solution of this problem has not been attempted. Because of the large number of equations, it will indeed be a difficult one.

4. A NUMERICAL PROCEDURE FOR OBTAINING CONSTANT GROUP DELAY WITH EQUI RIPPLE ERROR AND EXAMPLES

There are many problems of an equiripple or pseudo equiripple nature which are difficult, if not nearly impossible, to formulate by the equiripple rational function of Chapter 2. Two such problems are considered in this chapter. The first is a logical extension of the constant group delay approximation undertaken in Section 3.4; the ripples are weighted such that they increase with frequency. The second is the determination of the pole locations of an all-pass network to compensate the non-constant group delay characteristics of a sharp cutoff filter.

The proposed method of solution for these two problems utilizes the rapid convergence properties of the Newton-Raphson iteration technique. The steps taken to obtain an equiripple or pseudo equiripple solution are outlined below:

- a. A crude solution is found in which the desired number of ripples are present. This solution depends upon the nature of the problem; in some cases it is easily obtained, in others it is more difficult.
- b. If there are N arbitrary unknown parameters available, then N equations, F_i , $i=1, \dots, N$ are formed by setting the N distances from the maxima or minima to a known reference equal to zero.
- c. The maxima and minima, if not known analytically, are obtained numerically by Newton's method.¹⁷
- d. If the reference mentioned in (b) is chosen appropriately, then after one iteration, using Newton-Raphson, the N equations are solved with sufficient accuracy that the reference can be shifted towards the desired ultimate solution. The above process is then repeated until the final reference value has been reached.

The two problems mentioned above will now be used as examples to illustrate this procedure.

Example (4.1):

The group delay can be made to approximate a constant over a larger interval by increasing the ripple factor. This increase, however, causes more

over and undershoots to be present (introduction). As a compromise it seems reasonable that the group delay should approximate a constant best where most of the spectral energy falls, or where the magnitude function is largest. With this type of weighted error the approximation interval would be greater, and there would be less distortion in the impulse response.

The problem is thus two fold, (1) to pick a desirable weighting function, and (2) to determine the pole locations of an all pole transfer function such that the peaks and valleys of the group delay fall on the boundary of this weighted error region.

One attractive weighting function is

$$(\ln \epsilon_2 / \epsilon_1) (\omega / \omega_1)^2 \quad (4.1)$$

$$\text{error} = \pm \epsilon_1 \epsilon$$

where

$$\epsilon_1 = \text{the zero frequency error}$$

$$\epsilon_2 = \text{the error at the end of the approximation interval } \omega_1.$$

With this weighting function, a typical group delay ($n = 7$) will be as shown in Figure 20. Obviously, a crude approximation to this problem is already known, i.e., the equiripple approximation developed in Section 3.4. The N pole locations can now be specified in terms of the N peak deviations occurring at $\omega_2, \dots, \omega_{N+1}$ by means of the N non-linear equations

$$F_i = \tau(\omega_{i+1}^2) - \left[1 + (-1)^{i+1} \epsilon_1 \epsilon^{\frac{(\ln \epsilon_2 / \epsilon_1) (\omega_{i+1})^2}{\omega_1^2}} \right] = 0 \quad (4.2)$$

for $i=1, \dots, N$.

These equations, in turn, are function of ϵ_1, ϵ_2 , and the pole locations. That F_i is a function of the poles can be seen from the expressions for the group delay.

$$\tau(\omega_j^2) = - \sum_{i=1}^N \frac{s_i}{\omega_j^2 + s_i^2} \quad (4.3)$$

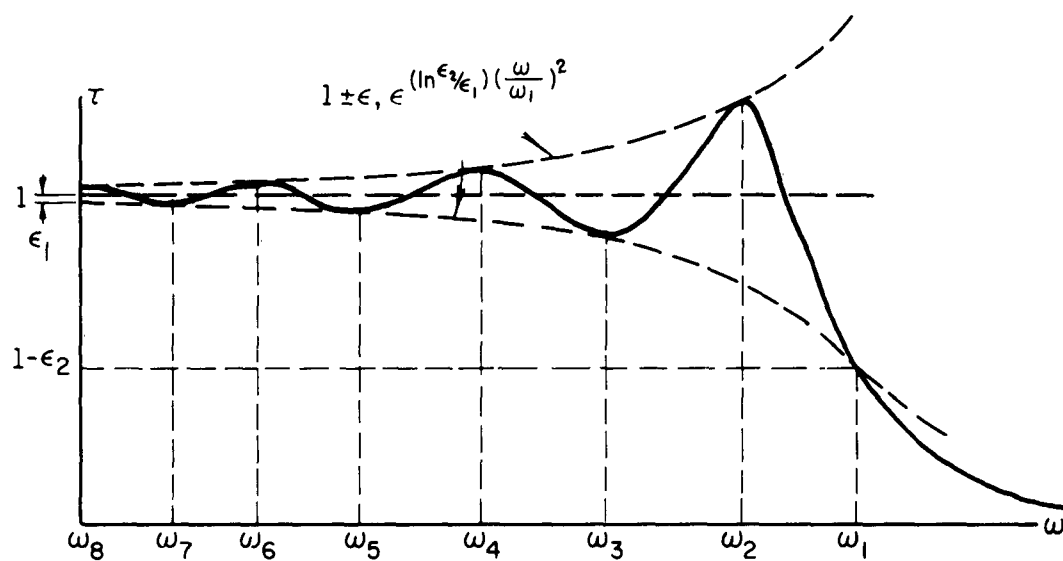


Figure 20. The group delay with an exponentially increasing error ripple
($N = 7$)

and the maxima and minima, the latter being zeros of

$$\frac{d \tau(\omega^2)}{d\omega^2} = \sum_{i=1}^N \frac{s_i}{(\omega^2 + s_i^2)^2} = 0. \quad (4.4)$$

An initial choice of the pole locations and the references ϵ_1 and ϵ_2 can be made such that the set F_1 is solved to a reasonable degree of accuracy in one iteration. A new ϵ_1 and ϵ_2 are now chosen and the process repeats until the desired solution results. Many factors dictate the number of iterations required to drift the poles into the neighborhood of the solution. Among these are: (a) the required overall change in ϵ_1 and ϵ_2 from the starting values, and (b) the pessimistic, or optimistic, choice of the incremental changes in ϵ_1 and ϵ_2 . A pessimistic choice requires many more iterations than necessary and an overly optimistic choice results in a solution outside the region of convergence, requiring a new start. Once ϵ_1 and ϵ_2 reach their final values, continued iteration will give the necessary exactitude to the pole locations.

A computer program was written to perform the above operations. Very few iterations were required for the final answers to be accurate to at least 8 significant figures.

A typical example, with $N = 10$, $\epsilon_1 = 0.01$, and $\epsilon_2 = 0.16$, has the pole distribution shown in Figure 21. For comparison, the initial pole distributions of the equiripple group delay filter with $\epsilon = 0.01$, and a Bessel filter are given also.

The effect of the weighted ripple approximation on the impulse response is best illustrated by a low order example. With $N = 3$, $\epsilon_1 = 0.005$ and $\epsilon_2 = 0.08$ the impulse response is computed and compared to that of the equiripple group delay ($\epsilon = 0.005$) as illustrated in Figure 22. Two differences are noted in the figure. One is that the symmetry is improved about the impulse's maximum due to the increased group delay approximation interval. The other, increased overshoot, is a decided disadvantage. Both of these differences, however, are small, and what is more, these differences become less noticeable for higher N .

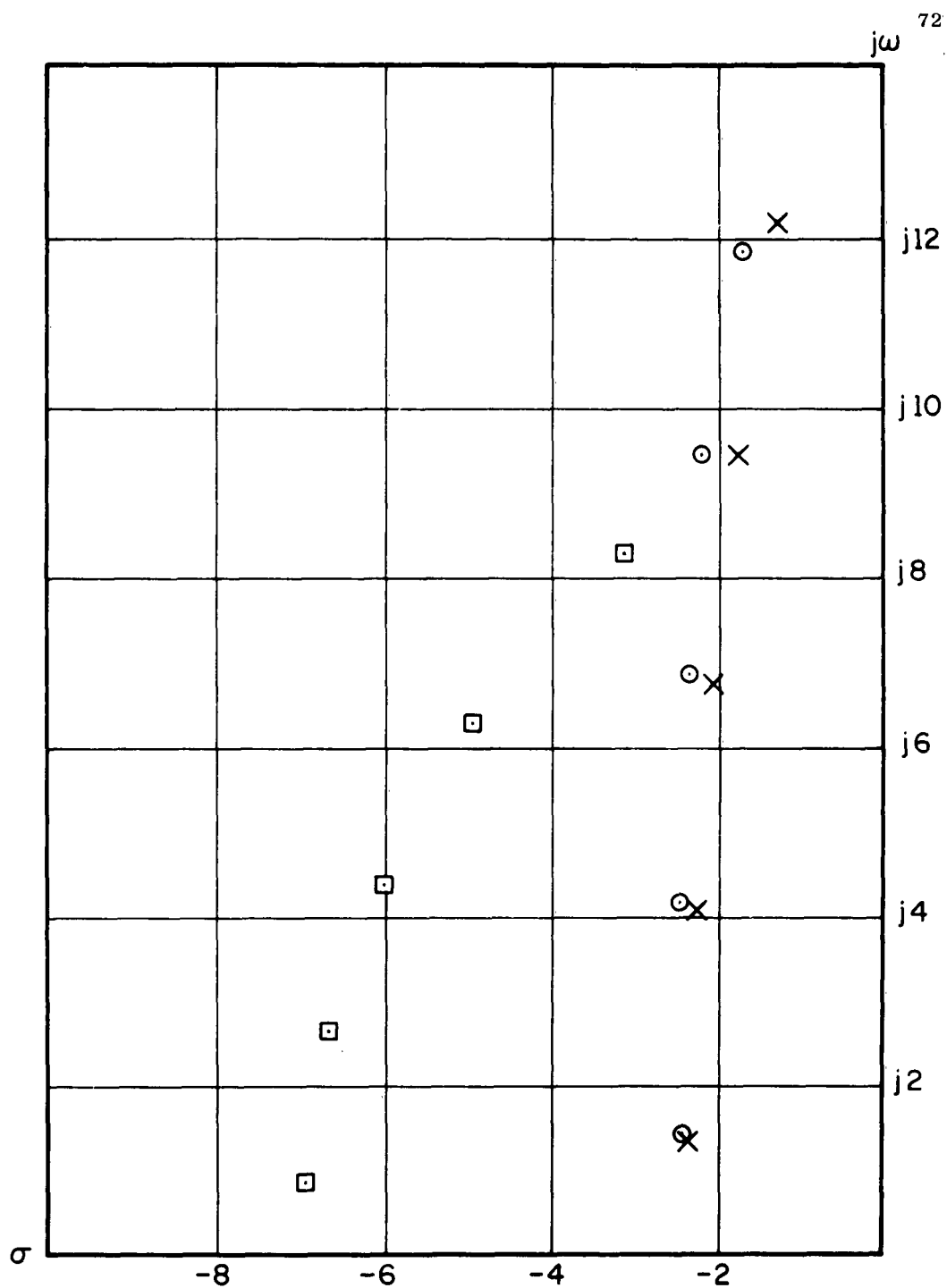


Figure 21. The upper half-plane pole distributions for x weighted ripple, $\epsilon_1 = 0.01$, $\epsilon_2 = 0.16$; ○ equiripple, $\epsilon = 0.01$; □ Bessel. All configurations unity delay

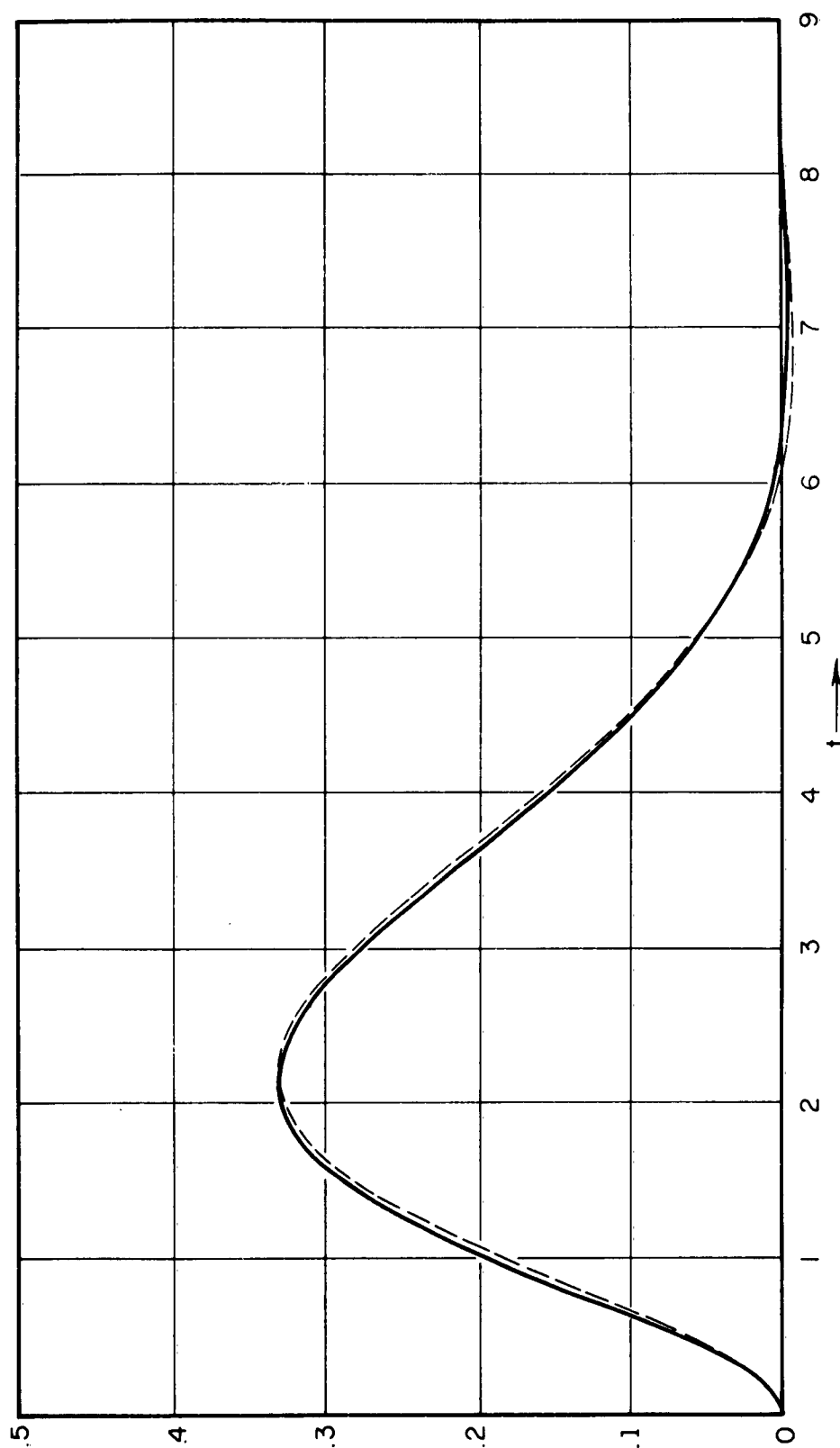


Figure 22. Comparison of the equiripple (solid) and the weighted ripple impulse response. The 6 db bandwidth is unity and $\epsilon = \epsilon_1 = 0.005$, $\epsilon_2 = 0.08$

Example (4.2):

There are many existing procedures for compensating the non-constant delay characteristics of sharp-cutoff filters.^{7,12,15,27} Foremost among these methods is Darlington's Chebyshev polynomial series approach¹⁰ which many authors have used.^{12,15,27} This method requires that the group delay be expanded in a Fourier series, and various dominant terms in the series are made negligible by an appropriate choice of all-pass network constants. This method has one disadvantage, i.e., the number of Fourier series terms contributing to the group delay error is unnecessarily large when the group delay of a sharp cutoff filter is being expanded over an interval that extends beyond the filter's cutoff.¹²

This example is used to illustrate a method wherein the poles of an all pass compensating network can be found to give an overall equiripple group delay characteristic to the low and all-pass filter combination. Furthermore, the delay can be made equiripple as far beyond cutoff as is desired.

Before the problem proper is started, one property of a minimum phase sharp cutoff filter needs to be investigated. Guillemin, using the Hilbert transform, has shown that the group delay of an ideal filter (as defined in Section 3.3) increases in a monotonic manner until it reaches a peak at or near the filter's cutoff. Beyond this point the group delay decreases monotonically to zero.¹⁴ This type of behavior is indeed observed in the classical Butterworth²⁴ and the low ripple Chebyshev filter (for larger ripples the group delay increases but in a non-monotonic manner). The group delay of a typical sharp cutoff lowpass filter is shown in Figure 23. In order that the overall filter - all-pass combination possess a constant group delay, the shaded area must be added by the all-pass (an all-pass giving negative group delay is unrealizable). An equiripple error solution to this problem can now be obtained in the following sequence:

- a. It is noted that the narrow peak occurring at the cutoff ω_c is due primarily to the pole in the low pass filter which is nearest the cutoff frequency, and also nearest the $j\omega$ -axis of the complex s -plane.
- b. An equally spaced pole distribution parallel to the $j\omega$ -axis will give an overall group delay characteristic that is pseudo

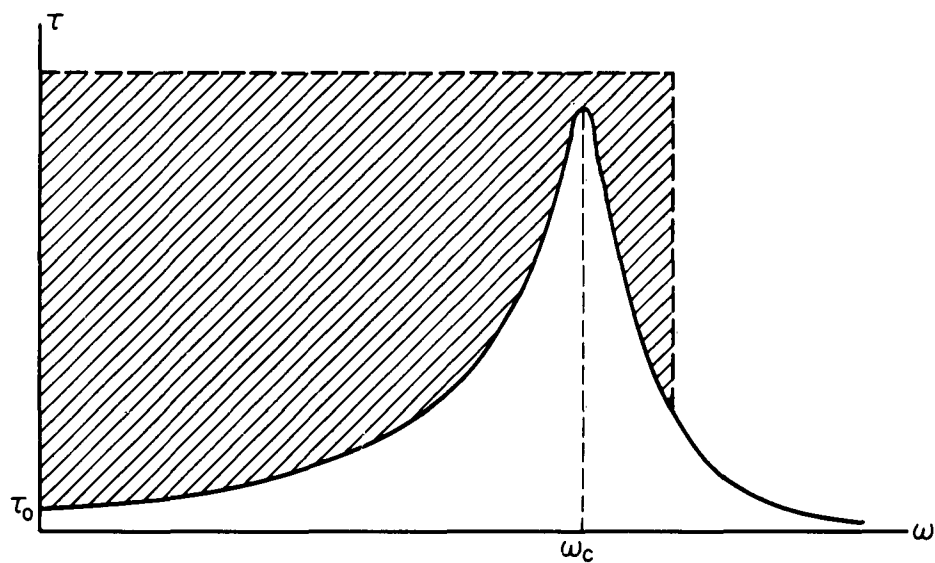


Figure 23. The group delay of a typical sharp cutoff filter. The shaded area is the amount of delay which must be added by an all-pass section

equiripple. The closer the poles are to the $j\omega$ -axis, the larger is the ripple.¹⁴

- c. If the pole distribution of (b) is chosen such that it includes the pole mentioned in (a), then an "almost" equiripple group delay can be achieved. The pole locations and the delay characteristics for this combination are shown in Figure 24.
- d. In the work that follows assume that an even number, N , of complex poles have been added to the fixed complex pole. The N equations

$$F_i = \tau(\omega_i) - \tau(\omega_{i+2}) = 0 \quad (4.5)$$

$$i = 1, \dots, N$$

will be of an even number and can be solved by Newton-Raphson to determine the exact locations of the added N poles to give an exact equiripple group delay. Once N and the fixed pole location have been specified, no further control exists over the ripple ϵ or the average delay τ_0 .

- e. The group delay for the above equiripple configuration can be written in the form

$$\tau(\omega^2) = -\frac{k}{2} \sum_{i=1}^N \frac{s_i}{\omega^2 + s_i} + \tau_{fp}(\omega^2) \quad (4.6)$$

where τ_{fp} is the group delay of the fixed complex pole. The value of k is one when the added poles are simple. Again using Newton-Raphson to solve (4.5), the value of k can be increased from one to two in about 5 single iteration steps. This makes the added poles double.

- f. The remaining poles of the low pass filter can now be included in the group delay to give

$$\tau = -\sum_{i=1}^N \frac{s_i}{\omega^2 + s_i} + \tau_{fp} + \frac{K}{2} \sum_{i=1}^{M-2} \frac{s_i}{\omega^2 + s_i} \quad (4.7)$$

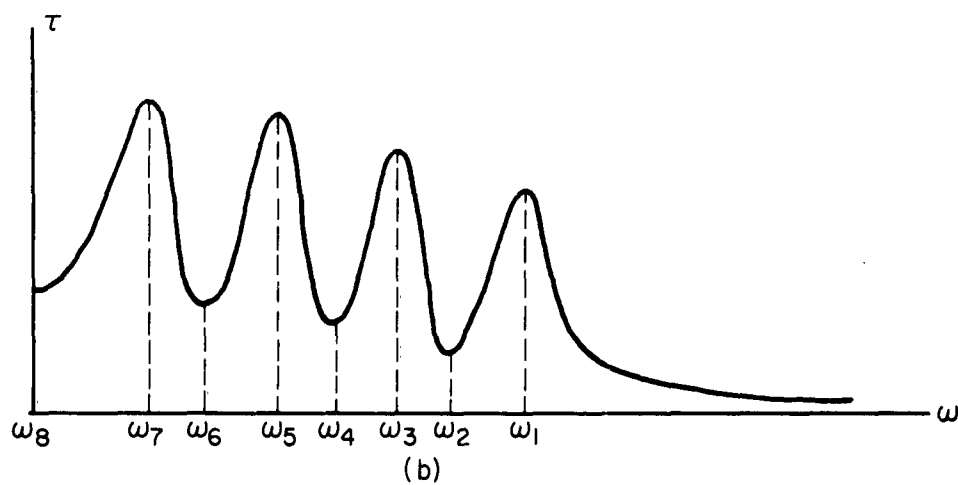
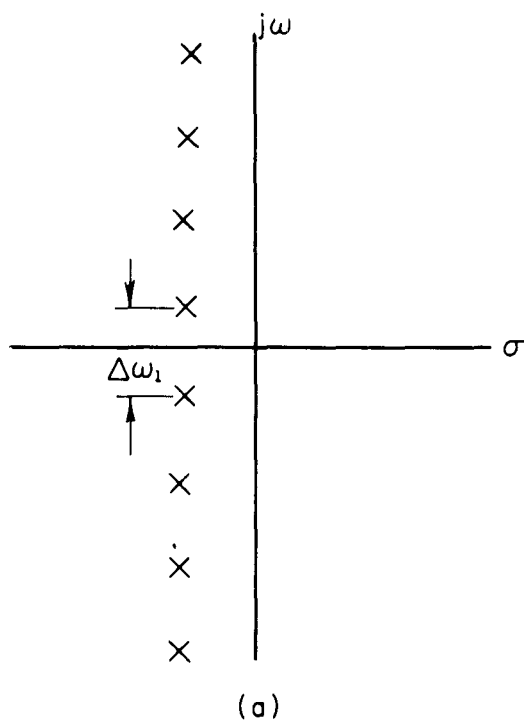


Figure 24. (a) A parallel pole distribution, and (b) its associated group delay

where M is the number of filter poles. The value of K can be gradually changes from zero to one using the Newton-Raphson technique to solve (4.5). This process shifts the added pole locations such that the equiripple group delay is still maintained.

- g. The compensation is completed by noting that the group delay of one double order pole at $s = -\sigma_1 + j\omega_1$ is the same as one simple pole at $s = -\sigma_1 + j\omega_1$ together with one zero located at $s = \sigma_1 + j\omega_1$. This latter combination gives the all-pass network.

For convenience two classifications are given to the compensator poles. Type A is given to the complex pole pairs whose imaginary part is less than the cutoff frequency. Similarly, Type B is given to those poles with an imaginary part greater than the cutoff frequency.

From the digital computer program written to perform the indicated steps, two different all-pass compensators were obtained from the Butterworth filters of degrees 4 through 8. One compensated the group delay up to the cutoff frequency (one Type A section); the other compensated the group delay up to and beyond the cutoff (one Type A and one Type B section). The corresponding pole locations are given in Table 4. The impulse responses of the Butterworth filters (degrees 4-8) with (a) no compensation, (b) one A section and (c) one A and one B section are shown in Figures 25, 26, and 27 respectively. The Butterworth filter of degree 3 with Type A compensation will not work by this method because the dominant pole is not close enough to the imaginary axis and the addition of another complex pole pair results in a delay without ripples. This particular case has already been solved by O'Meara with a single real pole all-pass section¹⁹ and needs no further discussion.

Several conclusions which in turn agree with the idealized time domain studies of Bangert³ can be drawn from an inspection of Figures 25-27. Some of these conclusions are:

- a. Type A compensation doesn't necessarily decrease the magnitude of the main side lobe, it merely puts the large side lobe in front of the main pulse instead of behind it.

TABLE 4

The following table gives the pole-zero locations of the compensating all-pass network and the filter being compensated (τ_0 = low frequency mean delay, $\Delta\tau$ = peak deviation from τ_0).

Compensator pole-zeros		Filter poles		Comments
real part	imag. part	real part	imag. part	
$\pm .688654$	± 2.30652	-1.00000 - .500000	0.0 $\pm .866025$	Butterworth, N=3, 1B section $\tau_0=2.936$, $\Delta\tau=.4611$
$\pm .769347$	$\pm .420376$	- .382683 - .923790	$\pm .923880$ $\pm .382684$	Butterworth, N=4, 1A section $\tau_0=6.630$, $\Delta\tau=.01338$
$\pm .587409$ $\pm .386700$	$\pm .371540$ ± 1.42347	- .382683 - .923879	$\pm .923880$ $\pm .382684$	Butterworth, N=4, 1A and 1B section, $\tau_0=8.270$, $\Delta\tau=.08257$
$\pm .500331$ $\pm .329566$	$\pm .374832$ ± 1.43535	-1.00000 - .309017 - .809017	0.0 $\pm .951056$ $\pm .587785$	Butterworth, N=5, 1A and 1B section, $\tau_0=9.180$, $\Delta\tau=.2155$
$\pm .644130$	$\pm .411718$	-1.00000 - .309017 - .809017	0.0 $\pm .951056$ $\pm .587785$	Butterworth, N=5, 1A section $\tau_0=7.711$, $\Delta\tau=.06597$
$\pm .546426$	$\pm .404039$	- .258819 - .965926 - .707107	$\pm .965926$ $\pm .258819$ $\pm .707107$	Butterworth, N=6, 1A section $\tau_0=8.770$, $\Delta\tau=.1735$
$\pm .433629$ $\pm .287087$	$\pm .374337$ ± 1.43698	- .258819 - .965926 - .707107	$\pm .965926$ $\pm .258819$ $\pm .707107$	Butterworth, N=6, 1A and 1B section, $\tau_0=10.09$, $\Delta\tau=.4094$
$\pm .469585$	$\pm .397092$	-1.00000 - .222521 - .623490 - .900968	0.0 $\pm .974928$ $\pm .781831$ $\pm .433885$	Butterworth, N=7, 1A section $\tau_0=9.803$, $\Delta\tau=.3422$
$\pm .380686$ $\pm .254003$	$\pm .371971$ ± 1.43496	- .222521 - .623490 - .900968 -1.0000	$\pm .974928$ $\pm .781831$ $\pm .433885$ 0.0	Butterworth, N=7, 1A and 1B section, $\tau_0=11.01$, $\Delta\tau=.6574$
$\pm .408057$	$\pm .390432$	- .195090 - .831470 - .980787 - .555570	$\pm .980785$ $\pm .555570$ $\pm .195090$ $\pm .831470$	Butterworth, N=8, 1A section $\tau_0=10.81$, $\Delta\tau=.5715$
$\pm .337518$ $\pm .227328$	$\pm .368411$ ± 1.43155	- .195090 - .831470 - .980787 - .555570	$\pm .980785$ $\pm .555570$ $\pm .195090$ $\pm .831470$	Butterworth, N=8, 1A and 1B section, $\tau_0=11.92$, $\Delta\tau=.9525$
$\pm .198711$ $\pm .179438$ $\pm .197993$ $\pm .129761$	$\pm .164874$ $\pm .795079$ $\pm .469677$ ± 1.20926	- .111963 - .293123 - .362320	± 1.01156 $\pm .625177$ 0.0	Chebyshev, 1/2 db ripple, N=5, 3A and 1B sections $\tau_0=21.37$, $\Delta\tau=.7671$

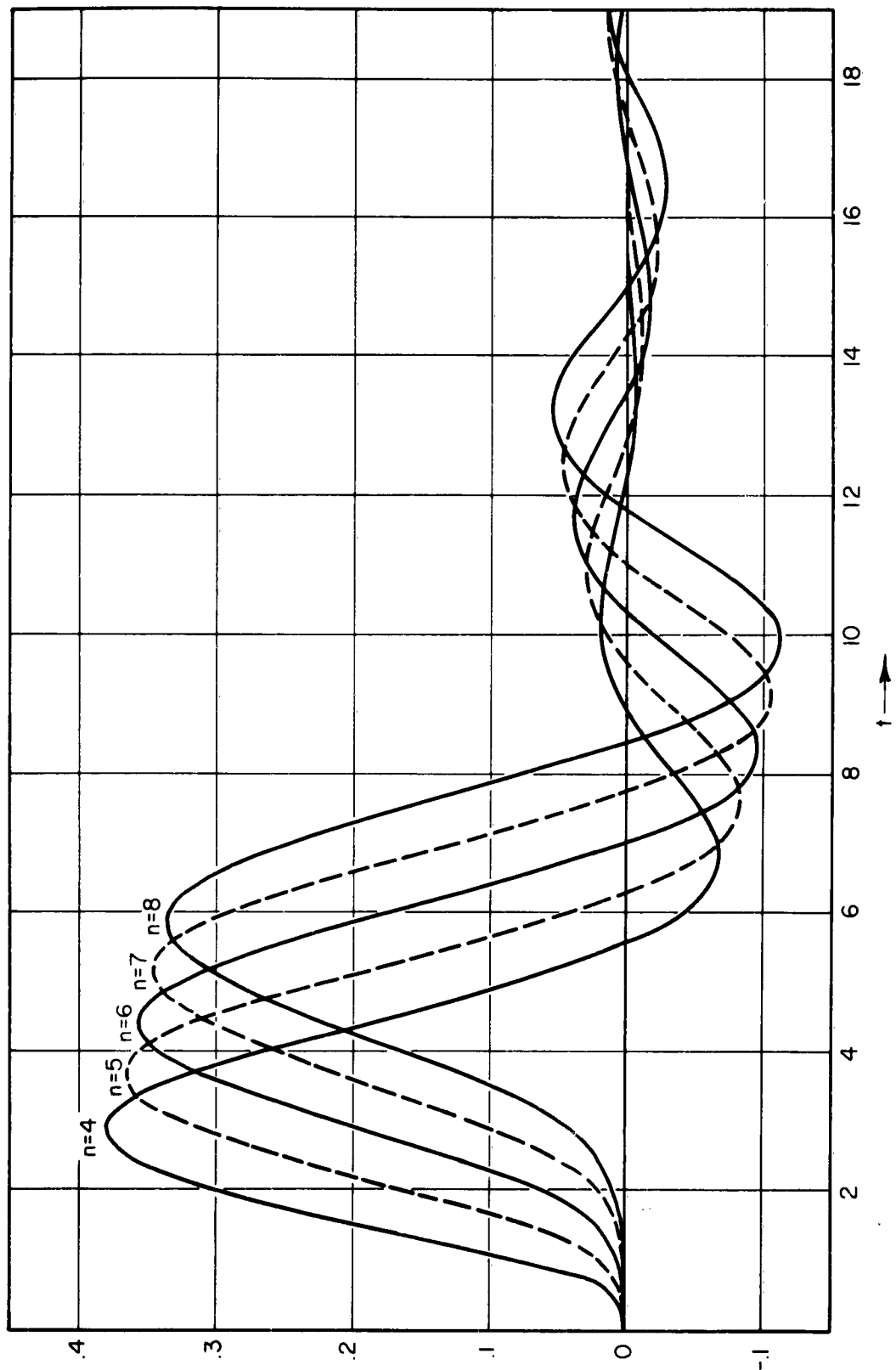


Figure 25. Impulse response of the Butterworth filter

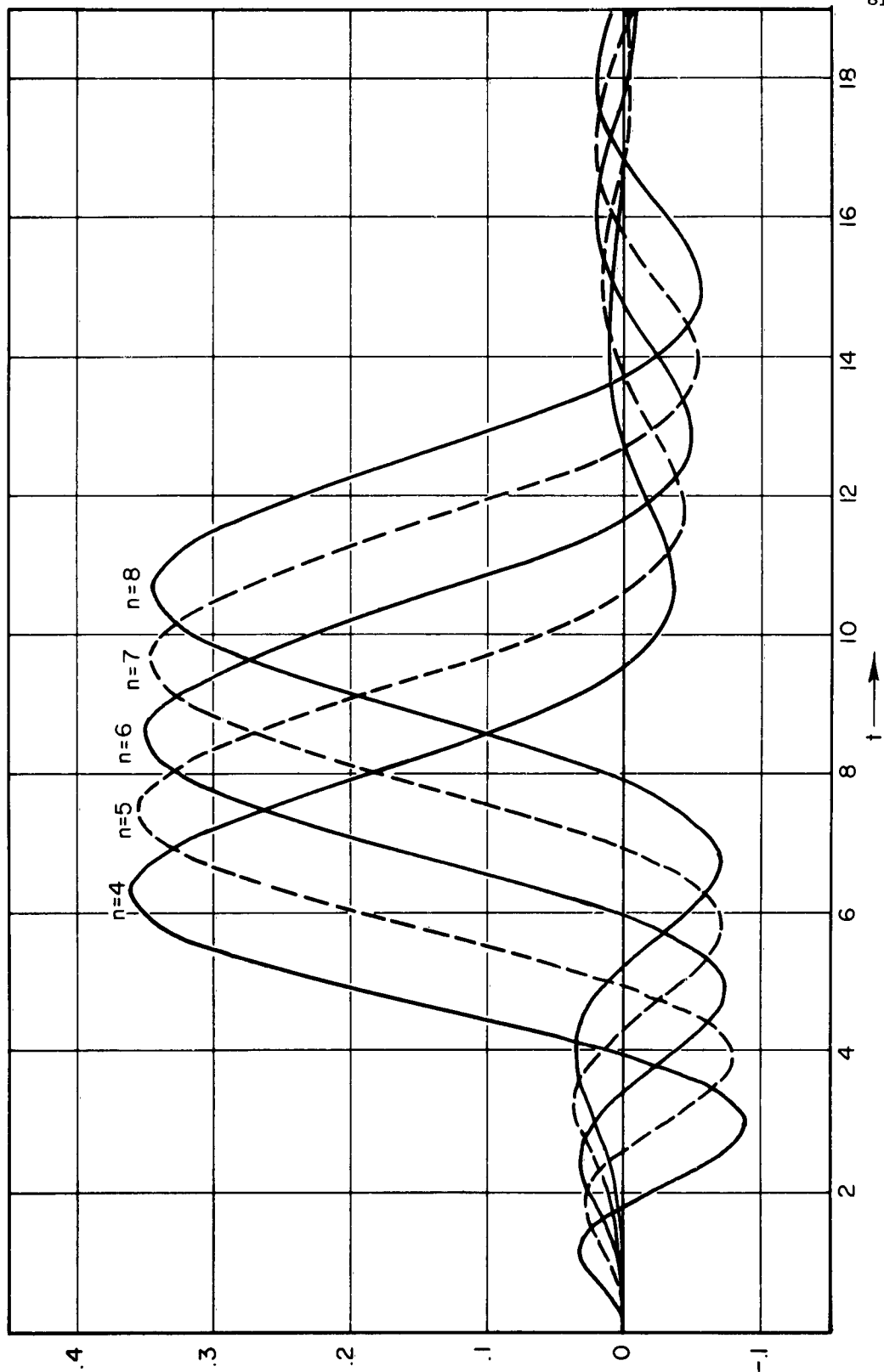


Figure 26. Impulse response of the Butterworth filter with 1 A section of delay compensation

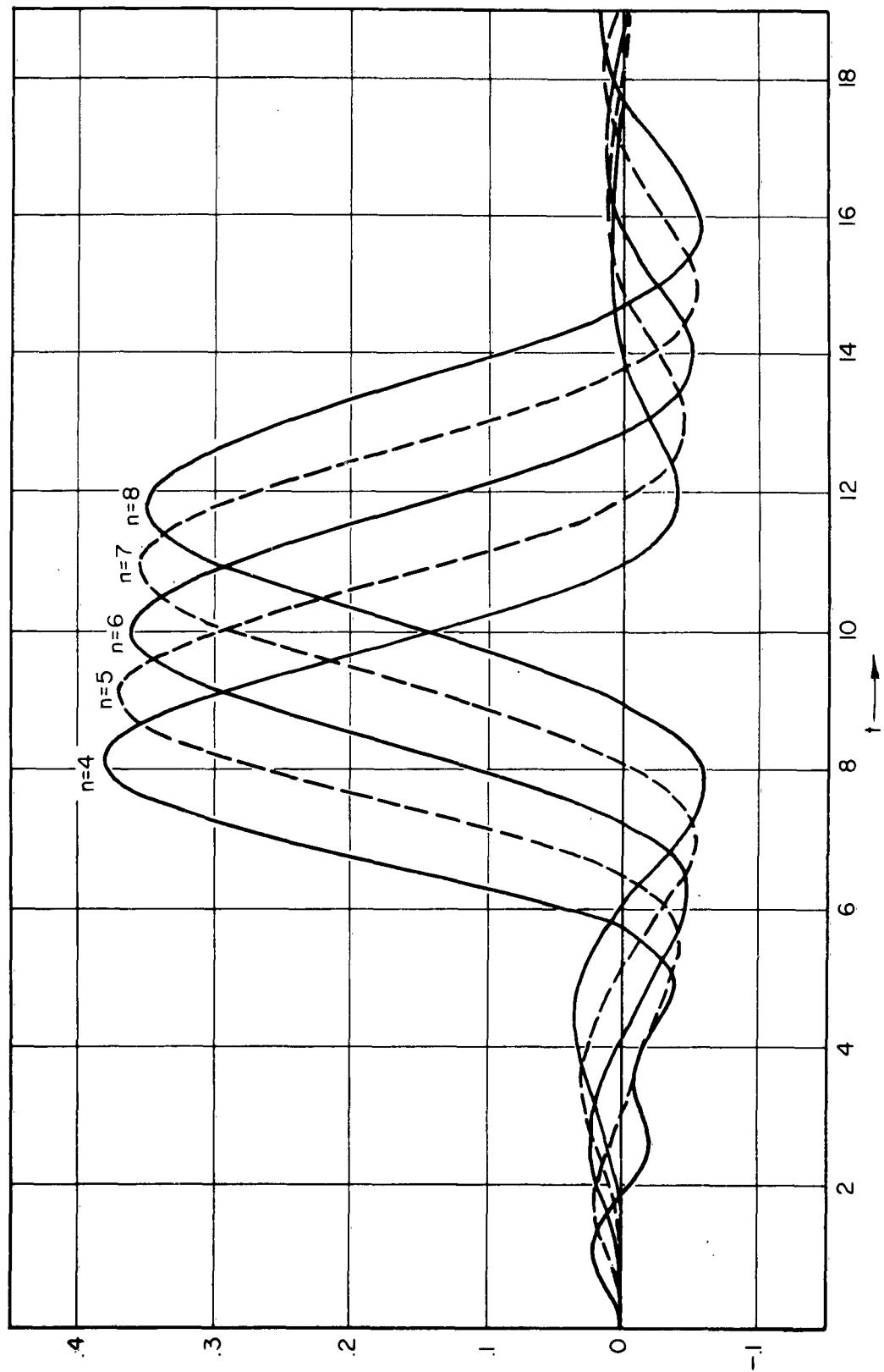


Figure 27. Impulse response of the Butterworth filter. 1 A and 1 B section of delay compensation are used

- b. The time from the excitation to the time the pulse has decayed to a small value is independent of the delay compensation.
- c. With Type A and B sections, the main pulse and the two large sidelobes are very close to $\sin x/x$.
- d. The distortion terms predicted in the introduction are of the form (assuming 100% efficiency)

$$\text{distortion} = \pm (-1)^m \frac{n}{4m} \epsilon \left[.35 \frac{\sin(t - \tau_o + 2\tau_o \frac{m}{n})}{(t - \tau_o + 2\tau_o \frac{m}{n})} \right] \quad (4.8)$$

Using the 8th degree Butterworth filter with one A and one B section the above is ($m = 6$, $n = 16$, $\tau_o = 11.9$ and $\epsilon = .08$)

$$\text{distortion} = \pm .0186 \frac{\sin(t - \tau_o + 8.9)}{(t - \tau_o + 8.9)} \quad (4.9)$$

Without actually plotting the sum of the distortion term and $.35 \sin(t - \tau_o)/(t - \tau_o)$, it can be seen that this will give a very close picture of Figure 27 (curve $n = 8$), since not only are the sign and the multiplying constant of the distortion term about correct, but also the magnitude of the delay shifts. From these results it is observed that the distortion terms so arrange themselves that the physical system does not display an anticipatory effect.

One more example of compensation will be given. This is a 5th order Chebyshev filter (1/2 db ripple) compensated with three A sections and one B section. The impulse response is shown in Figure 28. Indeed, for this example the output is very close to $\sin x/x$ for a considerable time on both sides of the main lobe.

Other filters up to 8th order have been compensated by this method. One difficulty was encountered with large ripple Chebyshev filters (2 db ripple). The group delay of this filter is not a monotonically increasing function for frequencies less than cutoff. Several small local minima and maxima are present. This necessitates an incremental change in K of (4.7) to be very small. Otherwise the computer program will evaluate the wrong minimax and hang up. Because of this, considerable computer time is required (about 4 minutes on the CDC 1604 for a sixth order filter with 5 or 6 section of compensation).

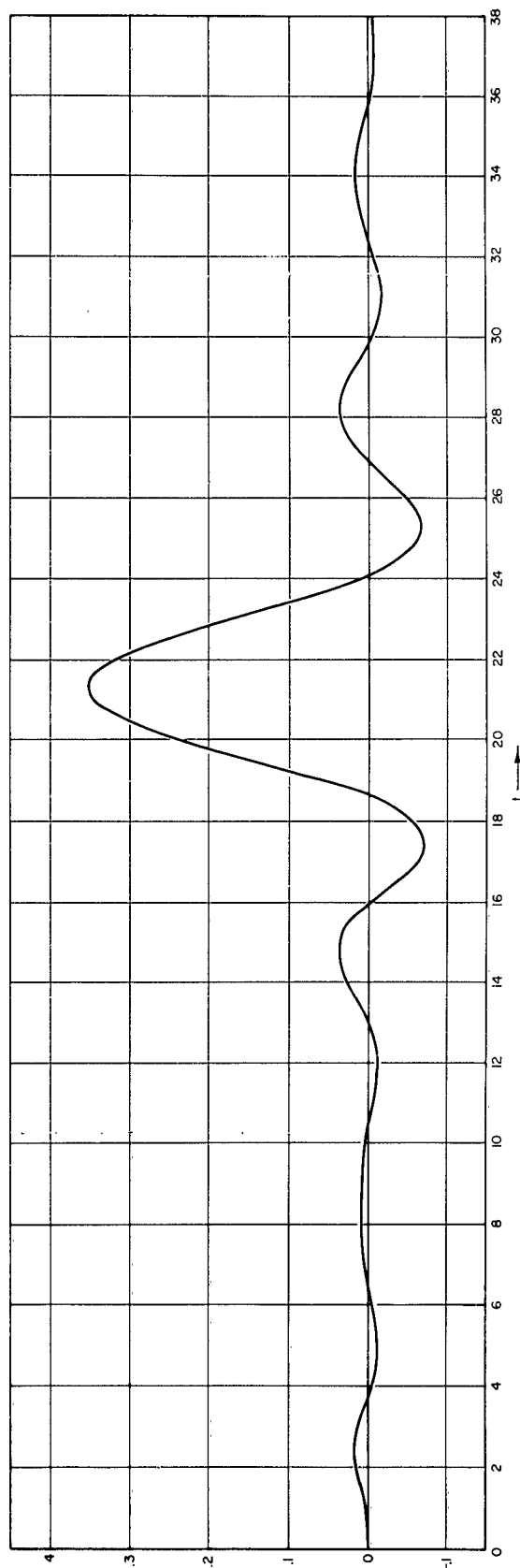


Figure 28. Impulse response of a Chebyshev filter $n = 5$, $1/2$ db ripple, with 3 A and 1 B section of delay compensation; $\tau_a = 21.374$, $\Delta\tau = .7671$, $\gamma = 3.58\%$

Problems of the preceding type can be worked much faster if interpolation schemes are used. As a parameter, K in (4.7) for example, is changed the past pole locations are remembered. From these, linear or parabolic interpolation can be used to estimate future locations. This method has been used on problems of the above type with very favorable results.

In summary, this chapter has made use of the digital computer's ability to perform the many iterations required in going from a rough guess to a useful or desirable solution. Indeed, by modifying Example (4.2) slightly, the equiripple group delay problem could have been solved.

5. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

A rational function which approximates a polynomial in an equiripple manner has been developed. The limitations of the Chebyshev rational function of Helman and Bennett, mentioned in Chapter 1, have been overcome. Some differences in these two functions are: the old form has arbitrary pole locations and the new one has arbitrary error maxima and minima; no transformations are involved in the new rational function, hence, the coefficients of the numerator and denominator are simple algebraic combinations of the maxima and minima error frequencies. This is in strong contrast to Bennett's form which depends upon a radical in a transformation. Consequently, the numerator of his rational function is cluttered with radicals, the even powers of which give a polynomial and, hopefully, the odd powers cancel.

Several examples were given to illustrate the use and flexibility of the new rational function. Some examples were used which have been treated using the old rational function; namely, the group delay and magnitude squared function approximations. These problems were adequately handled by the new rational function. In addition, several problems were undertaken which were handled easily by the new rational function. These in turn would be difficult or impossible to solve with the old rational function. Examples of this latter type are given in the sections on even and odd network function parts approximations.

It can be concluded that the new rational function can be a useful tool in approximating a low order polynomial to give an equiripple error. This is especially true when less than the maximum number of ripple is desired, or when the placement of the maxima and minima points is important.

As stated in Chapter 1, there are many problems which cannot be adequately handled by either equiripple error or rational function. However, many of these problems are of a type that can be solved using iterative techniques with the aid of a digital computer. A numerical scheme has been suggested to accomplish this latter goal. The method depends upon an initial guess which is already in itself a crude solution to the problem. This in many cases is a disadvantage.

There are many new and as yet unsolved problems which have been brought into focus by this work. Some of these are:

1. The non-linear simultaneous equations for the linearly increasing group delay need to be investigated and solved.
2. In this work it was assumed that various coefficients of the numerator could be set equal to zero. Necessary and sufficient conditions are needed to say whether this is always possible, or if not, when.
3. An extension of the developed theory to include the approximation of piecewise smooth polynomials would be useful.
4. Several degrees of freedom are used in making a transfer function realizable by a symmetrical network. The degree of freedom not utilized can be used to satisfy other constraints. One useful constraint on these freedoms would be to require an equiripple group delay solution, (however, the maximum number of ripples would not be present).

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His academic training began at Ricks College, Rexburg, Idaho in 1950. Two years later he transferred to the Brigham Young University where in 1955 he was in the first graduating class of two from the newly organized 5 year electrical engineering school. Two years later he received a Master's degree in electrical engineering from the University of Utah where he was a teaching fellow for one year and the recipient of a University Scholarship Grant for the remaining period.

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In addition to the above mentioned teaching positions he was an instructor at the Brigham Young University from 1957 to 1959. He has also been a special instructor for the two summers of 1959 and 1960. The former was at North American Aviation, Autonetics, Downey, California and the latter at the U.S. Navy Bureau of Ships, Washington, D.C.